Finite lattice RG-transformation for the 2D Ising model

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Abstract

We discuss a finite lattice approximation of an RG-transformation for the 2D Ising model on the square lattice. A 4×4 lattices is blocked to a 2×2 lattice. We restrict the discussion to the case of a vanishing external field. General features of an RG-transformation can be discussed at this example. Dispite the simplicity of the approximation the numerical results for the RG-exponent y_t compares quite well with the exact result.

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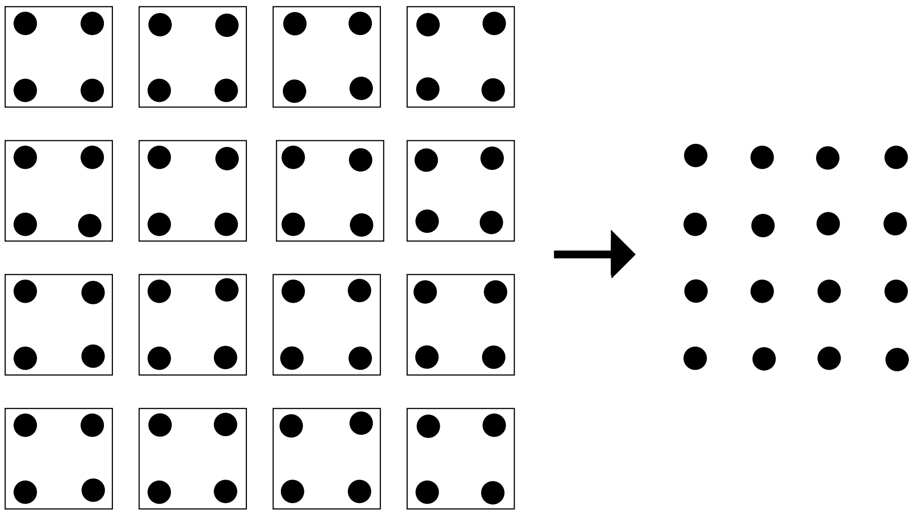

Figure 1: Blocking of a 8×8 lattice to a 4×4 lattice.

1 The RG-transformation

We consider the Ising model on the square lattice. In the following we study an blocking with the scale $b = 2$. The block-spin S_X takes the values -1 or 1. We use the majority rule to assign the value of the block-spin. (i.e. S_X is equal to the sign of the sum of the spins in the block.) Since the number of sites in a block is even, the sum of the spins in the block might vanish. In this case S_X is set to -1 or 1 with equal probability.

Let us define the transformation more precisely:
The sites of the original lattice are denoted by $x = (x_1, x_2)$, where $x_1, x_2 \in \{0, 1, ..., l-1\}$. The sites of the blocked lattice are denoted by $X = (X_1, X_2)$, where $X_1, X_2 \in \{0, 1, ..., L-1\}$. The blocking scale is b. The lattice size transforms as $L = l/b$. Block-sites and sites of the original lattice are related by

$$X(x) = (X_1(x_1), X_2(x_2)) = (\mathrm{int}(x_1/b), \mathrm{int}(x_2/b)) \quad . \tag{1}$$

The blockspin transformation is determined by the so called blocking kernel $T(S, s)$ which is a function of the blocked spins S and the original spins s. $T(S, s)$ can be interpreted as the probability to assign the block-spin configuration S to the configuration s. In general, the transformation kernel should satisfy the following conditions:

$$0 \leq T(S, s) \leq 1 \tag{2}$$

and

$$\sum_S T(S, s) = 1 \tag{3}$$

In addition we require that the kernel respects the symmetries of the model that is studied and that the blocking is local, i.e. that S_X is determined by the s_x in the block X (and by spins in the neighbourhood).

The Hamiltonian $H'(S)$ is defined by

$$\exp(-H'(S)) \ := \ \sum_s T(S,s) \ \exp(-H(s)) \tag{4}$$

Note that this definition only applies to finite lattices. In order to define a thermodynamic limit, we first have to extract the coupling constants. This can be achived by parametrizing the Hamiltonian like

$$H'(S) = \sum_\alpha K_\alpha H_\alpha(S) \tag{5}$$

where e.g.

$$H_2 = -\sum_\mu \sum_X S_X S_{X+\hat{\mu}} \tag{6}$$

is the interaction that we already know from the standard Ising model. In addition we have iteraction between spins with larger distances. There can also be interactions of more than two spins. In the simplest case the interaction of four spins with minimal distance:

$$H_4 = -\sum_X S_X S_{X+(1,0)} S_{X+(0,1)} S_{X+(1,1)} \tag{7}$$

In the presence of an external field also interactions of an odd number of spins are allowed. Note that there is no well defined order of the interaction terms. A general assumption is that the terms become less important as the distance between the spins increases.

In the following we take the scale factor $b = 2$ and study the transformation kernel

$$T(S,s) = \prod_X t(S_X, \sum_{x \in X} s_x) \tag{8}$$

with

$$\begin{aligned}
t(\ \ 1, \ \ 4) &= 1 \\
t(-1, \ \ 4) &= 0 \\
t(\ \ 1, \ \ 2) &= 1 \\
t(-1, \ \ 2) &= 0 \\
t(\ \ 1, \ \ 0) &= 0.5 \\
t(-1, \ \ 0) &= 0.5 \\
t(\ \ 1, -2) &= 0 \\
t(-1, -2) &= 1 \\
t(\ \ 1, -4) &= 0 \\
t(-1, -4) &= 1
\end{aligned} \tag{9}$$

We restrict our numerical study to the blocking of a 4×4 lattice to a 2×2 lattice. We use periodic boundary conditions. We also restrict our study to the case of a vanishing external field. Therefore on the 2×2 lattice there are only three distinct terms of the Hamiltonian:
the interaction of nearest neighbours, the interaction of next to nearest neighbours and the interaction of all four spins.

$$
\begin{aligned}
H = \; & - \; K_1 \sum_X S_X (S_{X+(1,0)} + S_{X+(0,1)}) \\
& - \; K_2 \sum_X S_X (S_{X+(1,1)} + S_{X+(1,-1)}) \\
& - \; K_3 \sum_X S_X S_{X+(1,0)} S_{X+(0,1)} S_{X+(1,1)} \\
& - \; const
\end{aligned}
\tag{10}
$$

All $2^{16} = 65536$ configurations on the 4^2 lattice can be easly generated on the computer. Therefore we can directly implement eq. (4) as a Fortran program.

On the 2×2 lattice there are $2^4 = 16$ configurations. For symmetry reasons only 4 classes of configurations with distinct Blotzmann factor remain:

C_1	C_2	C_3	C_4
$+\;+$ $+\;+$	$-\;+$ $+\;+$	$+\;+$ $-\;-$	$+\;-$ $-\;+$
$-\;-$ $-\;-$	$+\;-$ $+\;+$	$+\;-$ $+\;-$	$-\;+$ $+\;-$
	$+\;+$ $-\;+$	$-\;-$ $+\;+$	
	$+\;+$ $+\;-$	$-\;+$ $-\;+$	
	$+\;-$ $-\;-$		
	$-\;+$ $-\;-$		
	$-\;-$ $+\;-$		
	$-\;-$ $-\;+$		

Hence the evaluation of eq. (4) gives us the four numbers $H'(C_1)$, $H'(C_2)$, $H'(C_3)$ and $H'(C_4)$.

The terms of the Hamiltonian are

	C_1	C_2	C_3	C_4
H_1	8	0	0	-8
H_2	8	0	-8	8
H_3	4	-4	4	4

Note that periodic boundary conditions have to be taken into account. Therefore e.g. H_3 takes the value -4 or 4 and not -1 or 1.

Now we have four equations with the four variables K_1, K_2, K_3 and *const*. Their value can be obtained by solving the system of linear equations.

In order to iterate the transformation we put the new couplings back on the 4×4 lattice.

2 Numerical results

In figure 2 we show the projection of the flow of the couplings to the K_1, K_2 plane. The fixed point of the RG-transformation is

$$K^* = (0.299761200480, \quad 0.087094327205, \quad -0.00125863335) \tag{11}$$

The set of points in the coupling space that flows after infinitely many RG transformations into the fixed point is called the "critical surface". The intersection of the critical surface with the line $K = (K_1, 0, 0)$ is the critical coupling of the 2D Ising model on the square lattice with nearest neighbour interaction. The critical coupling of the 2D Ising model on the square lattice with nearest neighbour iteraction is

$$\beta_{c,approx} = 0.418197047162 \tag{12}$$

This value can be compared with the exact result

$$\beta_c = \frac{1}{2} \ln(\sqrt{2} + 1) = 0.4406867935098 \tag{13}$$

and the meanfield approximation $\beta_{c,MF} = 0.25$.

In order to obtain the critical exponent ν we have to study the linearized RG-transformation in the neighbourhood of the fixed point.

$$T_{\alpha\beta} = \frac{\partial K'_\alpha}{\partial K'_\beta}|_{K^*} \tag{14}$$

From finite differences we get

$$T = \begin{pmatrix} 1.358964 & 1.555979 & 0.602007 \\ 0.434221 & 0.749081 & 0.194741 \\ -0.004499 & -0.009853 & 0.131374 \end{pmatrix}$$

Note that T is not a symmetric matrix!

In order to obtain the critical exponent ν we have to compute the eigenvalues of the left eigenvectors of T:

$$\sum_\alpha \Phi^i_\alpha T_{\alpha\beta} = \lambda^i \Phi^i_\beta \tag{15}$$

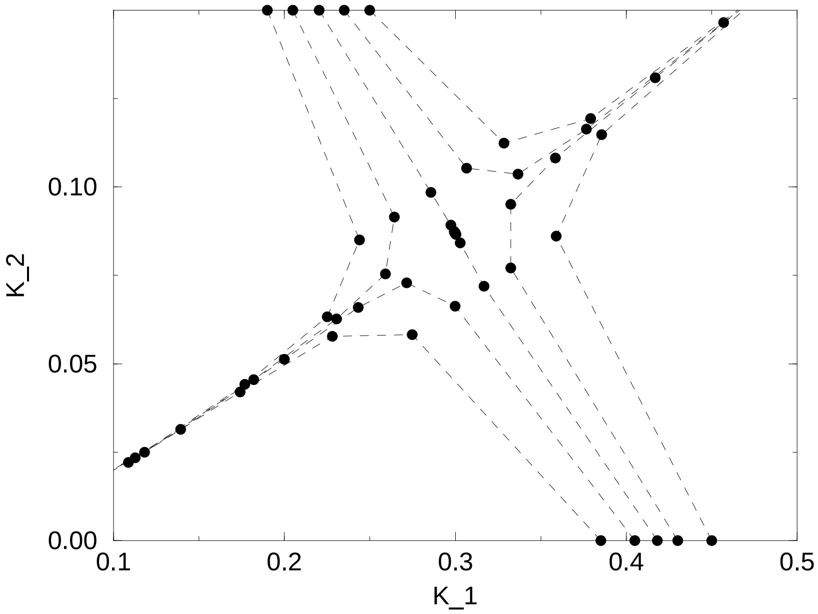

Figure 2: Flow of the couplings K_1 and K_2. K_3 is not displayed.

The results is

$$\begin{array}{lll} \lambda_1 = 1.928122 & , & \Phi^1 = (0.605106, 0.796143, 0.289033) \\ \lambda_2 = 0.178873 & , & \Phi^2 = (-0.409055, 1.104918, -0.654347) \\ \lambda_3 = 0.132424 & , & \Phi^3 = (-0.181359, 0.613127, 9.733203) \end{array} \tag{16}$$

Note that the eigenvectors of T are not orthogonal to each other!
The RG-exponents are defined by:

$$\lambda_i = b^{y_i} \tag{17}$$

We get

$$\frac{1}{\nu} = y_t = y_1 = 0.947196 \ , \ y_2 = -2.482992 \ , \ y_3 = -2.916763 \tag{18}$$

Discussion:
We have obtained one relevant RG-exponent. This qualitative result agrees with the exact result. Also the numerical value $y_t = 0.947196$ compares quite well with the exact result $1/\nu = y_t = 1$. Note that mean-field predicts $\nu = 0.5$!

3 Exercises:

1) Add an external field to the problem. How many new couplings have to be added? Compute the exponent y_h.
2) Study generalizations of the block-transformation:

$$t(u, w) = \frac{\exp(\kappa u w)}{\exp(\kappa w) + \exp(-\kappa w)} \tag{19}$$

The transformation kernel $T(S, s)$ is defined as in eq. (8). Note that in the limit $\kappa \to \infty$ we recover the blocking kernel as discussed above.

References

[1] J. Cardy: Scaling and Renormalization in Statistical Physics, Cambridge University Press 1996.