10 Quantum fields in curved space

10.1 Quantum fields in flat space

- We will focus on a real scalar field φ with

$$S = \int d^4x \, \mathcal{L} \quad ; \quad \mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2$$

(lagrange density)

with wave equation

$$(\Box - m^2)\varphi = 0 .$$

- We can view this as a classical mechanical system with many variables, $\varphi(t,\bar{x})$, labelled by the continuous index $\bar{x}$.

- The canonical momenta are

$$\pi(t,\bar{x}) = \frac{\delta}{\delta\dot{\varphi}(t,\bar{x})} L \quad ; \quad L = \int d^3\bar{x} \, \mathcal{L}$$

- It is easy to see that

$$\pi = \frac{\partial}{\partial\dot{\varphi}} \mathcal{L} = \dot{\varphi} .$$

- The Hamilton-fct. is

$$H = \int d^3\bar{x} \, \pi\dot{\varphi} - L = \int d^3\bar{x} \left(\pi\dot{\varphi} - \mathcal{L}\right)$$

$$H = \int d^3\bar{x} \, \underbrace{\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\bar{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2\right)}_{\mathcal{H} \text{ is the Hamilton density}}$$

- A set of solutions is easily found:

$$\varphi(x) = \varphi_0 e^{ikx} = \varphi_0 e^{-i\omega t + i\bar{k}\bar{x}}$$

with $k = (\omega, \bar{k})$ and $\omega^2 = \bar{k}^2 + m^2$ and $\omega > 0$.

- It is convenient to define scalar product on the space of solutions by

$$(\varphi_1, \varphi_2) = -i \int_\Sigma (\varphi_1 \dot{\varphi}_2^* - \varphi_2^* \dot{\varphi}_1)\, d^3x$$

(Σ is some constant-t hypersurface)

- With respect to this inner product, we want to find an orthonormal set of solutions. In the flat case considered here this is easy:

$$f_{\bar{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega}}\, e^{ikx}$$

satisfy $(f_{\bar{k}_1}, f_{\bar{k}_2}) = \delta^3(\bar{k}_1 - \bar{k}_2)$.

- The fcts. $f_{\bar{k}}(x)^*$ are also solutions. They satisfy

$$(f_{\bar{k}_1}, f_{\bar{k}_2}^*) = 0 \quad \text{and} \quad (f_{\bar{k}_1}^*, f_{\bar{k}_2}^*) = -\delta^3(\bar{k}_1 - \bar{k}_2).$$

- Together, $f_{\bar{k}}$ & $f_{\bar{k}}^*$ form a complete set.
- $f_{\bar{k}}$ & $f_{\bar{k}}^*$ are called positive- and negative-frequency solutions (because of the sign-change in the exponent $e^{-i\omega t}$ by "$*$").
- Canonical quantization proceeds by promoting φ & π to operators and demanding

$$[\varphi(t,\bar{x}), \varphi(t,\bar{x}')] = 0$$
$$[\pi(t,\bar{x}), \pi(t,\bar{x}')] = 0$$
$$[\varphi(t,\bar{x}), \pi(t,\bar{x}')] = i\delta^3(\bar{x} - \bar{x}')$$

(In fact, it suffices to demand this at some fixed t. The validity for all t then follows from EOM's in the Heisenberg picture. $\varphi(t,\bar{x})$ and $\pi(t,\bar{x})$ have to be understood as

Heisenberg-picture operators.)

- We can expand $\varphi(t,\bar{x})$ in the previous solutions:

$$\varphi(t,\bar{x}) = \int d^3\bar{k} \left\{ a_{\bar{k}} f_{\bar{k}}(t,\bar{x}) + a^{+}_{\bar{k}} f^{*}_{\bar{k}}(t,\bar{x}) \right\}$$

operators functions

- The previous commutation relations then imply

$$[a_{\bar{k}}, a_{\bar{k}'}] = 0 \; ; \; [a^{+}_{\bar{k}}, a^{+}_{\bar{k}'}] = 0 \; ; \; [a_{\bar{k}}, a^{+}_{\bar{k}'}] = \delta^3(\bar{k}-\bar{k}')$$

To derive this, first note that

$$a_{\bar{k}} = (\varphi, f_{\bar{k}}) = -i\int d^3\bar{x}\, (\varphi \dot{f}^{*}_{\bar{k}} - f^{*}_{\bar{k}}\pi)$$

(φ, π are hermitian!)

$$a^{+}_{\bar{k}'} = \quad \cdots \quad = i\int d^3\bar{x}' (\varphi \dot{f}_{\bar{k}} - f_{\bar{k}}\pi)$$

$$[a_{\bar{k}}, a^{+}_{\bar{k}'}] = \int d^3\bar{x} \int d^3\bar{x}' (-i)\delta^3(\bar{x}-\bar{x}') \Big[\dot{f}^{*}_{\bar{k}}(t,\bar{x}) f_{\bar{k}'}(t,\bar{x}') - f^{*}_{\bar{k}}(t,\bar{x}) \dot{f}_{\bar{k}'}(t,\bar{x}') \Big]$$

$$= -i\int d^3\bar{x} \left[f_{\bar{k}'}(t,\bar{x}) \dot{f}^{*}_{\bar{k}}(t,\bar{x}) - f^{*}_{\bar{k}}(t,\bar{x}) \dot{f}_{\bar{k}'}(t,\bar{x}) \right]$$

$$= (f_{\bar{k}'}, f_{\bar{k}}) = \delta^3(\bar{k}-\bar{k}')\,,$$ as required.

(and analogously for the other commutation relations)

- Thus, we have found an infinite set of harmonic-oscillator-algebras, labelled by $\bar{k}$. Introducing a vacuum state $|0\rangle$, with

$$a_{\bar{k}}|0\rangle = 0 \quad \text{for any } \bar{k},$$

we can define whole Hilbert space as the space spanned by the Basis

$$\underbrace{|(n_1,\bar{k}_1);\cdots;(n_\ell,\bar{k}_\ell)\rangle}_{\text{state, in which the harmonic osc. } \bar{k}_1 \text{ is excited to the level } n_1, \cdots \text{ etc.} \cdots} = \frac{1}{\sqrt{n_1!\cdots n_\ell!}}\left(a^+_{\bar{k}_1}\right)^{n_1}\cdots\left(a^+_{\bar{k}_\ell}\right)^{n_\ell}|0\rangle$$

normalization factor

- <u>Phys interpretation:</u> state, in which there are n_1 particles with momentum $\bar{k}_1$, ..., n_ℓ particles with momentum $\bar{k}_\ell$.

  (Since we started with one real scalar field, there is only one <u>type</u> of particle; each particle is its own antiparticle.)

- The above basis is called <u>Fock basis</u> and the corresponding Hilbert space is frequently called the <u>Fock space</u>.

- <u>Comment:</u> Even though we have broken Poincare symm. by choosing a frame in which to define H, our result is Poincare-symmetric in the following sense:

  - We can define quantum-mech. operators corresponding to the generators of the Poinc. group and use them to boost & rotate a given state. The resulting state will have the same # of particles (in particular $|0\rangle' = |0\rangle$), but with appropriately transformed momenta $\bar{k}_i'$ instead of $\bar{k}_i$.

  - This situation will be identical to what one would have found by quantizing in the appropriate different Lorentz frame from the very beginning.

- It is instructive to express the Hamiltonian through the creation & annihilation operators ($\rightarrow$ problems):

$$H = \frac{1}{2}\int d^3\bar{k}\cdot\omega(\bar{k})\left[a^+_{\bar{k}}a_{\bar{k}} + a_{\bar{k}}a^+_{\bar{k}}\right] \qquad \left(\text{recall: } \omega(\bar{k}) = \sqrt{\bar{k}^2+m^2}\right)$$

$$= \int d^3\bar{k}\,\omega(\bar{k})\left[n_{\bar{k}} + \frac{1}{2}\delta^3(\bar{0})\right]$$

"number operator" $n_{\bar{k}} \equiv a^+_{\bar{k}}a_{\bar{k}}$

comes from $[a_{\bar{k}}, a^+_{\bar{k}'}] = \delta^3(\bar{k}-\bar{k}')$.
The infinity of $\delta^3(\bar{0})$ is not problematic: it is just a manifestation of the infinite volume:

$\hookrightarrow$ Had we started in finite volume, the values of $\bar{k}$ would have been discreet and instead of $\delta^3(\bar{0})$ we would have found $V\cdot\delta_{ii}$ (where i labels the discreet momenta).

- What is much more serious is the diverging prefactor

$$\int d^3\bar{k}\,\omega(\bar{k}).$$

This is a true "UV divergence of QFT". It corresponds to adding up zero-point energies of infinitely many harmonic oscillators (with higher and higher frequencies). It can be removed by renormalization [adding a compensating classical piece]. However, the (by definition) finite difference of these two infinite contributions is naturally expected to be large ($\sim M_p^4$; since we expect $|\bar{k}|_{max} \sim M_p$). Experimentally, this energy density (the cosmological constant) is found to be $\sim (\text{meV})^4$. $\rightarrow$ "cosmol. constant problem".

10.2 Quantum fields in curved space

- Let
$$S = \int d^4x \sqrt{-g} \left(-\tfrac{1}{2} g^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) - \tfrac{1}{2} m^2 \varphi^2 \right)$$
on some Lorentzian manifold with metric $g_{\mu\nu}$. This corresponds to "minimal coupling" to gravity. (We ignore possible non-minimal terms $\sim R\varphi^2$ etc.)

- We consider the metric as fixed an quantize φ as usual:
$$\pi = \frac{\delta L}{\delta \dot{\varphi}} \quad ; \quad \begin{aligned} [\varphi(t,\bar{x}), \varphi(t,\bar{x}')] &= 0 \\ [\pi(t,\bar{x}), \pi(t,\bar{x}')] &= 0 \\ [\varphi(t,\bar{x}), \pi(t,\bar{x}')] &= \frac{i}{\sqrt{-g}} \delta^3(\bar{x}-\bar{x}') \end{aligned}$$
(necessary for covariance, as one can easily convince oneself by considering, e.g., $t \to \alpha t$ & $\bar{x} \to \beta \bar{x}$)

<u>Note</u>: Even though the above clearly relies on choosing a specific (timelike!) coordinate and calling it <u>time</u>, general covariance is respected. This should be intuitively clear from the fact that the crucial statement
$$[\varphi, \pi] \sim \delta^3(\cdot\cdot)$$
is essentially local (δ-fct. & equal times).
However, locally our manifold looks like Minkowski space and we already know that field quantization in Minkowski space is not affected different choices of t.

- As before, we consider the space of solutions, this time of the equ.

$$0 = (\Box - m^2)\varphi = (D_\mu \partial^\mu \varphi - m^2)\varphi = 0 .$$

- On this space, we define a scalar product

$$(\varphi_1, \varphi_2) = i \int_\Sigma \sqrt{\gamma}\, d^3x \, \underbrace{(\varphi_1 \partial_\mu \varphi_2^* - \varphi_2^* \partial_\mu \varphi_1)}\, n^\mu$$

Σ: spacelike hypersurface

$\sqrt{\gamma}$: induced metric on Σ

n^μ: unit normal vector for Σ

This is often written as $\varphi_1 \overleftrightarrow{\partial_\mu} \varphi_2^*$

(where $A \overleftrightarrow{\partial} B \equiv A \partial B - (\partial A) B$.)

[The scalar product is independent of the choice of Σ !]

- As before, one can always find a set of solutions, labelled by a (discrete or continuous) index i (instead of the wave number $\bar{k}$, which we don't have in general geometries):

$$f_i = f_i(x^\mu) \; ; \qquad (f_i, f_j) = \delta_{ij} \; ; \quad (f_i^*, f_j^*) = -\delta_{ij} \; ;$$

$$(f_i, f_j^*) = 0$$

- The (quantum) field can be expanded as

$$\varphi(x) = \sum_i \left(a_i f_i(x) + a_i^+ f_i^*(x) \right) .$$

a_i, a_i^+: operators; f_i, f_i^*: fcts.

- As before, the canonical commut. relations imply

$$[a_i, a_j] = 0 \; ; \quad [a_i^+, a_j^+] = 0 \; ; \quad [a_i, a_j^+] = \delta_{ij} .$$

- As before, we can define a Fock space on the basis of the vacuum $|0\rangle$ with $a_i|0\rangle = 0$ (any i).

- Clearly, different choices of complete sets exist, consider e.g.

$$\varphi = \sum_i (b_i g_i + b_i^+ g_i^*) \quad \text{with} \quad [b_i, b_j] = 0 \; ; \; [b_i^+, b_j^+] = 0$$

operators fcts.

$$[b_i, b_j^+] = \delta_{ij}.$$

(and, of course, $g_i \neq f_i$ in general).

- Now we have to distinguish two vacua, $|0_f\rangle$ (called $|0\rangle$ before) and $|0_g\rangle$:

$$a_i|0_f\rangle = 0 \; (\text{all } i) \quad ; \quad b_i|0_g\rangle = 0 \; (\text{all } i).$$

- So far, everything is still very similar to the flat case (where we also could choose different basis sets, e.g. by performing a Lorentz trf.). However, fcts. with positive frequency ($\partial_t f = -i\omega f$; $\omega > 0$) would transform among themselves, and so would fcts. with neg. frequency. This corresponds to

$$|0_f\rangle = |0_g\rangle \quad \text{for Lorentz trfs.}$$

- In general, there is no unique (not even up to Lorentz trfs.) time variable, no concept of pos./neg. frequency and hence

$$|0_f\rangle \neq |0_g\rangle \quad \text{in general.}$$

- To see this more explicitly, expand g_i in terms of f_i, f_i^*:

$$g_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} f_j^*).$$

(This is called a <u>Bogoliubov trf.</u>) By multiplic. with $f_k^{(*)}$ we find:

$$\alpha_{ij} = (g_i, f_j) \quad ; \quad \beta_{ij} = -(g_i, f_j^*).$$

- The above "Bogoliubov coefficients" can also be used to relate the relevant creation and annihilation operators:

$$\varphi = \sum_i \left(a_i f_i + a_i^+ f_i^*\right) = \sum_i \left(b_i g_i + b_i^+ g_i^*\right)$$

$$= \sum_{ij} \left(b_i \left\{ \alpha_{ij} f_j + \beta_{ij} f_j^* \right\} + b_i^+ \left\{ \alpha_{ij}^* f_j^* + \beta_{ij}^* f_j \right\} \right)$$

Comparing the coefficients of f_i, we find

$$a_i = \sum_j \left(\alpha_{ji} b_j + \beta_{ji}^* b_j^+ \right)$$

[An analogous calculation gives

$$b_i = \sum_j \left(\alpha_{ij}^* a_j - \beta_{ij}^* a_j^+ \right). \Big]$$

- The crucial physical point is that the Bogoliubov trf. does not only mix creation operators among themselves (& annih. ops. among themselves), but that it mixes creation with annihilation operators. Hence the vacuum of one description does not correspond to the vacuum of the other description.

- To see this quantitatively, let's calculate the particle number of the "g-system" in the vacuum of the "f-system":

$$\langle 0_f | n_{g_i} | 0_f \rangle = \langle 0_f | b_i^+ b_i | 0_f \rangle$$

$$= \sum_{jk} \langle 0_f | \left(\alpha_{ij} a_j^+ - \beta_{ij} a_j \right) \left(\alpha_{ik}^* a_k - \beta_{ik}^* a_k^+ \right) | 0_f \rangle$$

$$= \sum_{jk} \beta_{ij} \beta_{ik}^* \langle 0_f | \underbrace{a_j a_k^+}_{a_k^+ a_j + \delta_{jk}} | 0_f \rangle = \underline{\underline{\sum_j |\beta_{ij}|^2}}$$

This is generically non-zero!

- Observational consequence:

Let's assume that there exists a timelike Killing vector such that g_i & g_i^* are positive and negative frequency modes with respect to the corresponding time variable (this also requires that the space-time is static). More specifically,

$$g_i \sim e^{-i\omega t} \quad \& \quad g_i^* \sim e^{i\omega t}.$$

For a detector (observer) following orbits of the Killing vector field, the eigentime will be proportional to this time variable and it will detect particles according to the particle number defined in the g-system.

- If we know that the phys. system is in the state $|0_f\rangle$, then the detector will see particles as described by

$$\langle 0_f | N_{g_i} | 0_f \rangle \neq 0 \quad \text{(see above)}.$$

This is the basic argument behind Hawking radiation and the (technically similar but simpler) Unruh effect to be discussed in more detail in the next chapter.