11 Unruh effect and Hawking radiation

11.1 Uniformly accelerating observer

- Unruh effect: A uniformly accelerating observer in Minkowski space detects thermal radiation

- For simplicity, we consider 2d Minkowski space: $ds^2 = -dt^2 + dx^2$.

- A uniformly accelerating trajectory is given by

$$t(\tau) = \alpha^{-1} \sinh(\alpha\tau) \quad ; \quad x(t) = \alpha^{-1} \cosh(\alpha\tau)$$

- 4-velocity: $u^\mu = (\cosh(\alpha\tau), \sinh(\alpha\tau))$

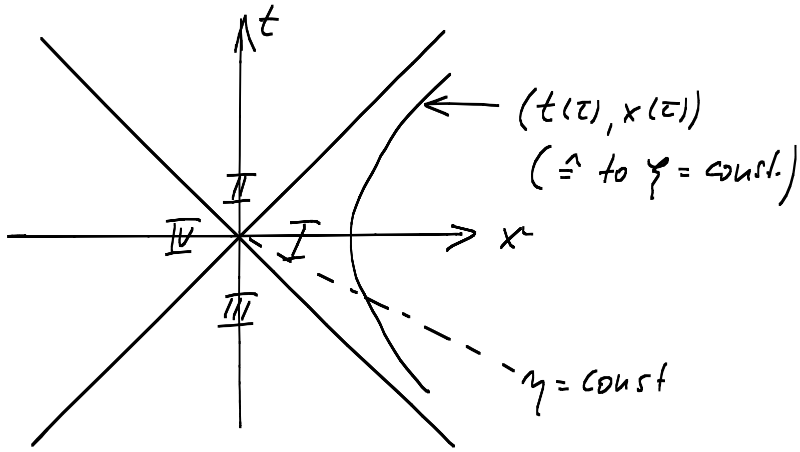
$u_\mu u^\mu = -\cosh^2(\alpha\tau) + \sinh^2(\alpha\tau) = -1$, so τ is indeed the eigentime

- 4-acceleration: $a^\mu = \dot{u}^\mu = (\alpha \sinh(\alpha\tau), \alpha \cosh(\alpha\tau))$

$$\sqrt{a_\mu a^\mu} = \sqrt{-\alpha^2 \sinh^2(\alpha\tau) + \alpha^2 \cosh^2(\alpha\tau)} = \alpha$$

This the constant acceleration.

- The trajectory satisfies the eq. $x^2 - t^2 = \alpha^{-2}$, i.e., it describes a hyperbola in region I of the following figure:



We can choose new coordinates (η, ξ) adapted to this trajectory:

$$t = a^{-1} e^{a\xi} \sinh(a\eta) \quad ; \quad x = a^{-1} e^{a\xi} \cosh(a\eta) .$$

- These coordinates only parameterize region I if $-\infty < \eta, \xi < \infty$.
- Our trajectory now reads
$$\eta(\tau) = \alpha a^{-1} \tau$$
$$\xi(\tau) = a^{-1} \ln(a\alpha^{-1}) .$$

[In particular, for $a = \alpha$, we simply get $\eta = \tau$, $\xi = 0$.]

- We can also express the metric through η & ξ:

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}\Big[- (d\xi \sinh(a\eta) + d\eta \cosh(a\eta))^2 + (d\xi \cosh(a\eta) + d\eta \sinh(a\eta))^2 \Big]$$

$$ds^2 = e^{2a\xi} (-d\eta^2 + d\xi^2)$$

Recall that, at an intermediate step of our discussion of Rindler spacetime we had

$$ds^2 = e^{v-u} \frac{1}{4} \left(-(du + dv)^2 + (dv - du)^2 \right) .$$

Thus, for $a = 1$ and with the identification

$$\xi = (v-u)/2 \quad ; \quad \eta = (v+u)/2 ,$$

we are simply dealing with one of the descriptions of Rindler spacetime. The time translation symm. of

$$ds^2 = -\tilde{x}^2 d\tilde{t}^2 + d\tilde{x}^2$$

corresponds to translations in η.

- For the so-called Rindler observer ($\hat{=}$ the uniformly accelerated observer in Minkowski space), the lines

$$t = \pm x \quad (x > 0)$$

play the role of the event horizon of the Schwarzschild solution for a static observer in the Schwarzschild geometry.

[e.g., no signal from beyond $t = x$ can ever reach any Rindler observer in region I.]

- In fact, it is easy to check that these "horizons" are actually Killing horizons of the Killing vector field $(\partial/\partial\eta)$.

11.2 Unruh effect

- We already know how to quantize a scalar field theory in Minkowski space using pos./neg. frequency modes of "t".

- In addition, we will now give a description based on the time variable η [The vacua will not agree, giving rise to radiation seen by a Rindler observer in empty Minkowski space.]

- Note first that $(\partial/\partial\eta)$ extends naturally to region IV:

$$\frac{\partial}{\partial\eta} = \frac{\partial t}{\partial\eta}\frac{\partial}{\partial t} + \frac{\partial x}{\partial\eta}\frac{\partial}{\partial x} = e^{a\xi}\left[\cosh(a\eta)\frac{\partial}{\partial t} + \sinh(a\eta)\frac{\partial}{\partial x}\right]$$

$$= a\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right),$$ which is defined in all of Minkowski space.

[In fact, it generates Lorentz boosts.]

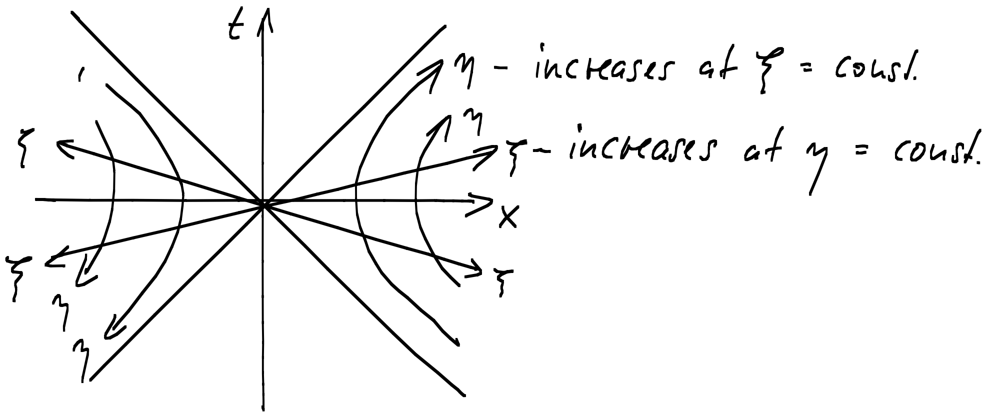
- Furthermore, we can parametrize region IV in analogy to region I by defining
$$t = -a^{-1} e^{a\xi} \sinh(a\eta) \; ; \; x = -a^{-1} e^{a\xi} \cosh(a\eta).$$

[In principle, we should give these coordinates new names to distinguish them from η, ξ of region I, but since many formulae will be very similar, it will be easier to abuse notation and avoid new names.]

- We will base our quantization on commut. relations at $t = 0$ ($\hat{=} (\eta = 0)$). We are in the lucky situation to have a timelike Killing vector field orthogonal to this spacelike hypersurface: $(\partial/\partial\eta)^\mu$ (both in I & IV).

[We do not need to worry that $(\partial/\partial\eta)^\mu (\partial/\partial\eta)_\mu = 0$ at $(x = 0, t = 0)$ – This is a set of measure zero.]

- This will allow us to define pos./neg. frequency modes on the basis of this Killing vector field. However, we have to be careful since $\partial/\partial\eta$ is future-directed in I and past-directed in IV. We want to define pos. frequency using a future-directed Killing field. Thus, we will have to treat I & IV separately (using $-\partial/\partial\eta$ in IV).



- To be explicit, we now focus on a massless scalar φ:

$$S = \int d^2x \sqrt{-g}\left(-\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)\right)$$

- Variation w.r.t. φ obviously gives the E.O.M.

$$\partial_\mu\left(\sqrt{-g}\,\partial^\mu\varphi\right) = 0 \qquad \left[\text{equivalent to } D_\mu\partial^\mu\varphi = 0\right]$$

- In Rindler coordinates: $ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$

$$\sqrt{-g} = e^{2a\xi}$$

$$\partial^{1,2} = e^{-2a\xi}\,\partial_{1,2} \quad ,$$

hence the E.O.M. is simply $\boxed{(-\partial_\eta^2 + \partial_\xi^2)\varphi = 0}$.

- Like in Minkowski space, the normalized plane-wave solutions read

$$g_k = \frac{1}{\sqrt{2\pi\cdot 2\omega}}\, e^{-i\omega\eta + ik\xi} \quad , \quad \omega = |k|$$

1-dim. "vector"

but we need to treat I & IV separately:

$$g_k^{(1)} = \begin{cases} (4\pi\omega)^{-1/2}\exp(-i\omega\eta + ik\xi) & \text{in I} \\ 0 & \text{in IV} \end{cases}$$

$$g_k^{(2)} = \begin{cases} 0 & \text{in I} \\ (4\pi\omega)^{-1/2}\exp(i\omega\eta + ik\xi) & \text{in IV} \end{cases}$$

- The required complete set of modes is formed by $g_k^{(1)}, g_k^{(2)}$ (both positive-frequency!) and their complex conjugates.

Check of normalization:

$$(\varphi_1, \varphi_2) = -i\int_\Sigma (\varphi_1 \overleftrightarrow{\partial_\mu} \varphi_2^*)\, n^\mu \sqrt{\gamma}\, d^1x$$

Σ: $\eta = 0$-surface

$n^\mu = (e^{-a\xi}, 0)$

since $g_{\mu\nu} n^\mu n^\nu = -e^{2a\xi}\left(e^{-a\xi}\right)^2 = -1$, as required.

$\sqrt{\gamma} = e^{a\xi}$; $\int d^1x \to \int d\xi$

$\Rightarrow (\varphi_1, \varphi_2) = -i\int d\xi \left(\varphi_1 \overset{\leftrightarrow}{\partial}_\eta \varphi_2^*\right)$ – just like 2d Mink.-space

$\Rightarrow$ our normalization of the modes $g_k^{(1)}, g_k^{(2)}$ is correct.

- We will not proof that these modes form a complete set in full Minkowski space, but we will at least show that they have an analytic extension in regions II & III (making our claim intuitive):

region I: $t = a^{-1} e^{a\xi} \sinh(a\eta)$; $x = a^{-1} e^{a\xi}\cosh(a\eta)$

$$\Rightarrow -t+x = a^{-1} e^{a\xi} e^{-a\eta}$$

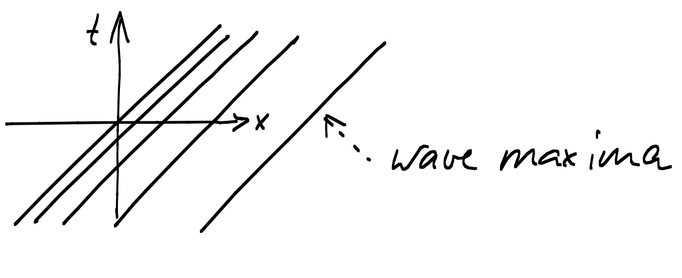
$$a(-t+x) = e^{-a(\eta-\xi)}$$

(analogous formulae are easily derived for $t+x$ in I & $-t+x$, $t+x$ in IV)

- This gives $g_k^{(1)} \sim e^{-i\omega\eta + ik\xi} \underset{\text{for } k>0}{=} e^{-i\omega(\eta-\xi)} =$

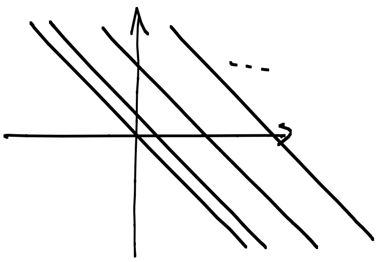
$$= e^{-i\omega\left(-\frac{1}{a}\ln(a(-t+x))\right)} = \left(a(-t+x)\right)^{i\omega/a}.$$

- This obviously can be used in I & III (but must be replaced by zero in II & IV)

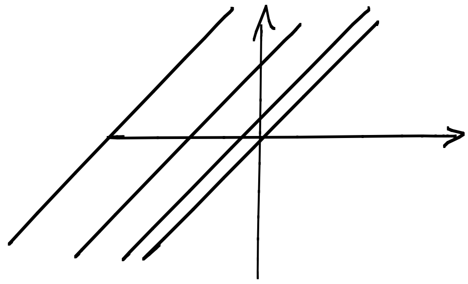


- Analogously:

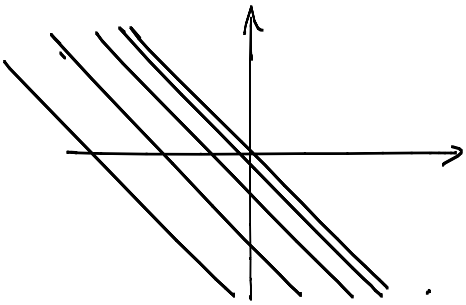
$g_k^{(1)}$: $(k<0)$



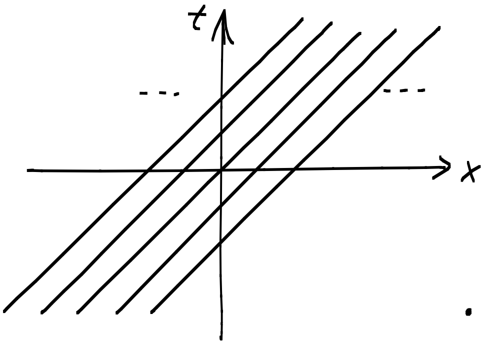
$g_k^{(2)}$: $(k>0)$



and finally: $g_k^{(2)}$ $(k<0)$



(For comparison, the wave maxima of a plane wave with $k>0$ in Minkowski coordinates are



.)

- We are now precisely in the situation described earlier in the context Bogoliubov tfs., i.e.

$$\varphi = \int dk \left(b_k^{(1)} g_k^{(1)} + b_k^{(1)+} g_k^{(1)*} + b_k^{(2)} g_k^{(2)} + b_k^{(2)+} g_k^{(2)*} \right)$$
(Rindler coords.)

and

$$\varphi = \int dk \left(a_k f_k + a_k^+ f_k^* \right) \quad \text{(Minkowski coordinates)}.$$

We could calculate the particle number seen by the Rindler observer by expanding the g's in terms of the f's (see above). However, there is a shortcut which consists in rearranging the g-basis such that part of it positive-frequency in the sense of Minkowski space:

- Consider $g_k^{(1)} = \frac{1}{\sqrt{4\pi\omega}} \exp\left(\frac{i\omega}{a} \ln\left(a(-t+x)\right)\right)$ non-zero at $t=0$; $x>0$

  and $g_{-k}^{(2)*} = \frac{1}{\sqrt{4\pi\omega}} \exp\left(\frac{i\omega}{a} \ln\left(a(t-x)\right)\right)$ non-zero at $t=0$; $x<0$

- Choosing the branch-cut of $\ln(-t+x)$ in the lower half of the complex $(-t+x)$-plane, we have

$$\ln(-t+x) \underset{\substack{\uparrow \\ -t+x<0}}{=} \ln\left((-1)\cdot(t-x)\right) = i\pi + \ln(t-x)$$

- Thus $e^{-\pi\omega/a} g_{-k}^{(2)*}$ is the analytic continuation of $g_k^{(1)}$

  and
$$h_k^{(1)} \equiv \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2)*}\right)$$

  [proper normalization factor]

  is an analytic fct., non-zero both at $x<0$ & $x>0$ $(t=0)$.

- Analogously,
$$h_k^{(2)} \equiv \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1)*}\right)$$

  provides a second, independent set of such fcts.

- Using the established normalization of $g_k^{(1)}, g_k^{(2)}$, it is easy to check (just by linearity of the scalar product) that $h_k^{(1)}, h_k^{(2)}$ are properly normalized & orthogonal, e.g.

$$\left(h_k^{(1)}, h_{k'}^{(1)}\right) = \cdots = \delta(k-k') \;, \text{ etc.}$$

- The crucial point is that $h_k^{(1)}$ is right-moving (depends on $-t+x$ only) and well-defined & bounded in the upper half of the $(-t+x)$-plane [for $k>0$]

- The same is true for $f_k \sim e^{-i\omega(t-x)}$ [for $k>0$].
- Similarly, $h_k^{(2)}$ are left-moving & well-defined and bounded in the lower half of the $(t+x)$-plane. [for $k>0$].
- The same is true for $f_k \sim e^{-i\omega(t+x)}$ [for $k<0$]
- Thus $h_k^{(1)}, h_k^{(2)}$ can be expressed in terms of f_k only (not using f_k^*!) [The signs of k are mixed, but that doesn't bother us.] *)
- This implies that $|0_f\rangle = |0_h\rangle$ (although $|0_f\rangle \neq |0_g\rangle$!).
- We now write
$$\varphi = \int dk \left(c_k^{(1)} h_k^{(1)} + c_k^{(1)+} h_k^{(1)*} + \cdots 1 \leftrightarrow 2 \cdots \right)$$

and
$$b_k^{(1)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\pi\omega/2a} c_k^{(1)} + e^{-\pi\omega/2a} c_{-k}^{(2)+} \right)$$

$$b_k^{(2)} = \quad \cdots \text{ same with } 1 \leftrightarrow 2 \cdots$$

[These are exactly the formula we had derived for the expansion of h's in terms of the g's. See also our general discussion of the Bogoliubov trf.]

- Finally, we have

$$\langle 0_f | n_g^{(1)} | 0_f \rangle = \langle 0_f | b_k^{(1)+} b_k^{(1)} | 0_f \rangle \qquad \left(\text{use that } |0_f\rangle = |0_h\rangle\right)$$

$$= \langle 0_f | \frac{1}{2\sinh\left(\frac{\pi\omega}{a}\right)} e^{-\pi\omega/a} c_{-k}^{(2)} c_{-k}^{(2)+} | 0_f \rangle$$

$$= \frac{e^{-\pi\omega/a}}{2\sinh\left(\frac{\pi\omega}{a}\right)}\,\delta(0) = \frac{1}{e^{2\pi\omega/a}-1}\,\delta(0) = \frac{1}{e^{\omega/T}-1}\,\delta(0).$$

This is just the Planck-spectrum (Bose-Einstein distribution) with $T = a/2\pi$. [$\delta(0)$ is simply the manifestation of the fact that this holds in all of infinite Mink.-space as seen by the Rindler observer.]

$\Rightarrow$ <u>Unruh-effect</u>: An observer moving with acceleration a through the Minkowski vacuum sees a thermal spectrum of (all!) particles with $T = a/2\pi$.

[Recall our Rindler coordinates are adapted to an observer moving with acceleration $\alpha = a$ along the trajectory $\xi = 0$. For him $\tau = \eta$ and he sees the particle # $n_g^{(1)}$ used above.]

<u>Note</u>: To really proof that the radiation is thermal one also needs to exclude correlations between particles. This can be done.

*) <u>Attempt to make the argument more precise:</u>

Let $h_k^{(1)} = \int dk' \left(\cdots f_k + \cdots f_k^* \right)$. For $k>0$, the l.h. side can be viewed as an analytic fct. of the complex variable $(-t+x)$ [not depending on $(t+x)$!], wich is well-def. & bounded at $\mathrm{Im}(-t+x) \to +\infty$. This is also true for f_k $(k>0)$. It is <u>not</u> true for

$$f_k^* \sim e^{i\omega(t-x)} \qquad (k>0)$$

which, viewed as an analytic fct. of $(-t+x)$, diverges at $\mathrm{Im}(-t+x) \to \infty$. Hence, f_k^* does not appear in the expansion.

Comment: • Our treatment follows S. Carroll's book, who follows the lecture notes of Ted Jacobson from Utrecht ("Introd. Lectures on BH Thermodyn."; accessible from his Web page).

• A description of the effect in the Minkowski frame is provided by W. Unruh and R. Wald (Phys. Rev. D, 29, p. 1047 (1984)):

The absorbtion (or emission) of a particle from the thermal bath by the detector in the Rindler frame corresponds to the emission of a Minkowski particle in the Minkowski frame. Thus, an accelerated detector emits particles.

11.3 Hawking radiation

(simplified treatment based on Unruh effect)

• Basic idea: A static observer in the outside region of a Schwarzschild BH is an observer moving along a trajectory of the time-translation Killing vector field ξ^μ. This observer is a uniformly accelerated observer which therefore sees thermal radiation analogously to the Unruh effect. In a certain limit, we will be able to make this correspondence precise and therefore derive Hawking radiation from the Unruh effect.

• We begin by calculating the acceleration of an observer moving along a timelike Killing vector field ξ^μ:

- The 4-velocity of this observer is $u^\mu = \xi^\mu / V$ with

$$V = \sqrt{-\xi_\mu \xi_\nu g^{\mu\nu}} = \sqrt{-\xi^\mu \xi^\nu g_{\mu\nu}}$$

- In flat space, $a_\mu = \underset{\substack{\uparrow \\ \text{eigentime}}}{\frac{d}{d\tau}} u_\mu = \frac{dx^\nu}{d\tau} \cdot \frac{\partial}{\partial x^\nu} u_\mu = u^\nu \partial_\nu u_\mu$.

- In curved space: $a_\mu = u^\nu D_\nu u_\mu$.

- Now consider $D_\mu(-V^2) = 2 \underbrace{(D_\mu \xi_\nu)}_{= -D_\nu \xi_\mu} \xi_\rho g^{\nu\rho}$

$= -D_\nu \xi_\mu$ using the Killing eq.

$$D_\mu \xi_\nu + D_\nu \xi_\mu = 0$$

$$\Rightarrow D_\mu(-V^2) = -2\xi^\nu D_\nu \xi_\mu$$

- We can now evaluate a_μ as

$$a_\mu = \frac{\xi^\nu}{V} D_\nu \left(\frac{\xi_\mu}{V} \right) = \frac{1}{V^2} \frac{1}{2} D_\mu V^2 - \underbrace{\frac{\xi^\nu \xi_\mu}{V^3} D_\nu V}$$

$$\Rightarrow a_\mu = \frac{1}{2} D_\mu \ln V^2 = \underline{\underline{\partial_\mu \ln V}}.$$

$= 0$ since the length of ξ^μ does not change along ξ^μ.

- For Schwarzschild BH's

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \quad ; \quad f(r) = 1 - \frac{2M}{r}$$

$$\xi^\mu = (1,0,0,0) \quad ; \quad V = \sqrt{f(r)}$$ ← (this is also the "redshift factor" as is obvious from our discussion of the gravit. redshift)

$$a_\mu = \partial_\mu \frac{1}{2} \ln\left(1 - \frac{2M}{r}\right) = \frac{1}{2\left(1 - \frac{2M}{r}\right)} \partial_\mu \left(-\frac{2M}{r} \right)$$

$$= \frac{M}{r^2\left(1 - \frac{2M}{r}\right)} \partial_\mu r \quad ; \quad \partial_\mu r = (0,1,0,0)$$

$$(\partial_\mu r)(\partial_\nu r) g^{\mu\nu} = f(r)$$

- $a = \sqrt{a_\mu a_\nu g^{\mu\nu}} = \dfrac{M}{r^2(1-\frac{2M}{r})} \cdot \sqrt{1-\frac{2M}{r}} = \dfrac{M}{r^2\sqrt{1-\frac{2M}{r}}}$.

- It is useful to recall that, actually, $M \to GM$ if we don't set $G = 1$. Thus,
$$a = \frac{r_h/2}{r^2\sqrt{1-r_h/r}} \quad ; \; r_h \text{ - horizon radius}$$

- For a static observer very close to the horizon (at r_1 with $r_1 - r_h \ll r_h$) we have $a_1 \gg r_h^{-1}$.

  The typical curvature scale in that region is r_h^{-1}. (There is no other scale in this problem!)

- Thus, we are dealing with a strongly accelerated observer in essentially flat space. He sees a temperature
$$T_1 = a_1/2\pi .$$

- If we slowly remove this observer to $r_2 > r_1$, he continues seeing the same particles (e.g. photons) but redshifted by V_1/V_2. He sees a temperature $T_2 = (V_1/V_2)\, T_1$. For $r_2 \to \infty$ we find
$$T = V_1 T_1 . \quad (V_2 \to 1 \text{ for } r_2 \to \infty).$$

  If we now let $r_1 \to r_h$, to make our original approx. better, we find
$$T = \lim_{r_1 \to r_h} V_1 T_1 = \lim_{r_1 \to r_h} \sqrt{1-\frac{r_h}{r_1}} \cdot \frac{1}{2\pi} \cdot \frac{r_h/2}{r_1^2\sqrt{1-\frac{r_h}{r_1}}}$$

$$\underline{\underline{T = \frac{1}{4\pi r_h}}}$$ This is the celebrated Hawking temperature.

(Due to this radiation, black holes eventually evaporate.)

11.4 What else you could have learned (from a better lecturer)

Black hole (Bekenstein-Hawking) entropy

Considering the BH as a thermodynamic object (e.g. in contact with a heat bath via its Hawking radiation), we can write the 1st law of thermodynamics as

$$dM = T\,dS. \quad (c=1)$$

$$T = \frac{1}{4\pi r_h} = \frac{1}{8\pi MG} \Rightarrow 8\pi G\, M dM = dS$$

integrate! $\Rightarrow \quad 4\pi G M^2 = S$

(assuming $S=0$ for $M=0$).

$$\Rightarrow \quad S = \frac{\pi}{G} r_h^2 = \frac{4\pi r_h^2}{4G} = \frac{A}{4G}$$

or, for $G=1$, $\underline{\underline{S = A/4}}$ (A - horizon area)

(The entropy counts microstates. If per Planck-scale piece of horizon area there is ~ one bit of information, this formula makes sense at the intuitive level.)

Brans-Dicke gravity

Theories of the type $S = \int d^4x \sqrt{g}\left(\varphi^2 R/2 - \alpha(\partial\varphi)^2 + \mathcal{L}_{SM}(g_{\mu\nu}, \psi)\right)$

are called Brans-Dicke or scalar-tensor theories. They can be brought to the so called "Einstein frame" by rescaling the metric:

$$g_{\mu\nu} = g'_{\mu\nu} \cdot (M^2/\varphi^2),$$

after which, however, φ couples to ψ in $\mathcal{L}_{SM}$. They are phenomenologically disfavoured (α has to be unnaturally large), but are interesting theoretically. They are viable if some small effect gives φ a mass ("stabilizes φ at $\langle\varphi\rangle = M$).

Kaluza-Klein-theories

If $S = \int d^5x \, \frac{1}{2} M_5^3 R$ and spacetime is $\mathbb{R}^4 \times S^1$ (with sufficiently small S^1 and time being non-compact), one can derive a "4d effective theory" which contains:

- 4d gravity $\quad (M_4^2 \sim M_5^3 \cdot (\text{radius of } S^1))$
- Brans-Dicke scalar (essentially g_{55})
- electrodynamics (essentially from $A_\mu \hat{=} g_{\mu 5}$).

This idea is central in many modern developments (supergravity, string theory, "large extra dimensions").

Spinors in general relativity

A massless Dirac spinor is described by

$$S = \int d^4x \, \bar{\psi} \, i \gamma^m \partial_m \psi \quad ; \quad \psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^4 \end{pmatrix}$$

γ^m – 4 4x4 matrices satisfying $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$.

To covariantize this, one really <u>needs</u> the vielbein formalism:

$$S = \int d^4x \sqrt{g} \, \bar{\psi} \, i \gamma^m e_m{}^\mu \underbrace{D_\mu \psi}_{\partial_\mu \psi_\alpha + \underbrace{(\omega_\mu)_\alpha{}^\beta}_{} \psi_\beta}$$

the so-called "spinor representation" of our $Lie(SO(1,3))$ matrix $(\omega_\mu)_a{}^b$.

The End