# 4 The Einstein Equation

## 4.1 Derivation of the Einstein equation

- We first focus on the gravity action:

$$\delta \int d^4x \sqrt{-g}\, R_{\mu\nu} g^{\mu\nu} = \int d^4x \left( (\delta\sqrt{-g}) R + \sqrt{-g}\, \delta R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} \right)$$

(1) $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\delta \det(g_{\mu\nu})$

- Consider a symmetric invertible matrix $A$.
- $\ln \det A = \operatorname{tr} \ln A$, which is obvious in the diagonalized form.
- $\delta(\ln \det A) = \delta(\operatorname{tr} \ln A) = \operatorname{tr}(\delta \ln A) = \operatorname{tr}(A^{-1}\delta A)$

recall that ln of a matrix is defined by the appropriate Taylor series, under the trace the order of terms is not important.

- also, $\delta \ln \det A = \frac{\delta \det A}{\det A} \Rightarrow \delta(\det A) = (\det A)\cdot \operatorname{tr}(A^{-1}\delta A)$

Thus $\delta g = g \cdot g^{\mu\nu}\delta g_{\mu\nu} = -g \cdot g_{\mu\nu}\delta g^{\mu\nu}$

since $0 = \delta(4) = \delta(g_{\mu\nu}g^{\mu\nu})$
$= \delta g_{\mu\nu} g^{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu}$.

$$\Rightarrow \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$$

(2) Let us first calculate $\delta R_{\mu\nu\rho}{}^{\sigma}$.

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = [D_\mu, D_\nu] v_\rho$$

$$\delta R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\delta D_\mu) D_\nu v_\rho + D_\mu (\delta D_\nu) v_\rho - \{\mu \leftrightarrow \nu\}$$

$$= \delta D_\mu \left(\partial_\nu v_\rho - \Gamma_{\nu\rho}{}^{\sigma} v_\sigma\right) - D_\mu \, \delta\Gamma_{\nu\rho}{}^{\sigma} v_\sigma - \{\mu \leftrightarrow \nu\}$$

(derivatives of $v$ will drop out)

$$= \underbrace{\delta\Gamma_{\mu\nu}{}^{\tau}\Gamma_{\tau\rho}{}^{\sigma} v_\sigma + \delta\Gamma_{\mu\rho}{}^{\tau}\Gamma_{\nu\tau}{}^{\sigma} v_\sigma} - D_\mu (\delta\Gamma_{\nu\rho}{}^{\sigma}) v_\sigma + \underbrace{\delta\Gamma_{\nu\rho}{}^{\sigma}\Gamma_{\mu\sigma}{}^{\tau} v_\tau} - \{\mu \leftrightarrow \nu\}$$

These terms will drop out after $\mu,\nu$ - antisymmetrization

$$\Rightarrow \delta R_{\mu\nu\rho}{}^{\sigma} = - D_\mu \delta\Gamma_{\nu\rho}{}^{\sigma} - \{\mu \leftrightarrow \nu\}$$

We are now ready to derive the final result, namely

$$g^{\mu\rho}\delta R_{\mu\rho} = g^{\mu\rho}\delta R_{\mu\nu\rho}{}^{\nu} = g^{\mu\rho}\left(-D_\mu \delta\Gamma_{\nu\rho}{}^{\nu} + D_\nu \delta\Gamma_{\mu\rho}{}^{\nu}\right) =$$

$$= D_\mu \left(-g^{\mu\rho}\delta\Gamma_{\nu\rho}{}^{\nu} + g^{\nu\rho}\delta\Gamma_{\nu\rho}{}^{\mu}\right) \equiv D_\mu v^\mu .$$

The object $v^\rho$ defined in this way is a vector field since $\delta\Gamma_{\mu\nu}{}^{\rho}$ is a tensor.

③ $R_{\mu\nu}\delta g^{\mu\nu}$ is already in the right form.

Combining ① .. ③ we get

$$\delta S_{grav.} = \frac{M^2}{2}\int d^4x \sqrt{-g}\left(D_\mu v^\mu + \left[R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right]\delta g^{\mu\nu}\right)$$

<u>Fact:</u> In analogy to $\int d^4x \, \partial_\mu v^\mu = 0$ (up to boundary terms), we have:

$$\underline{\int d^4x \sqrt{-g}\, D_\mu v^\mu = 0}$$

<u>Problem</u>: Demonstrate this fact. (As an intermediate step, derive that $D_\mu v^\mu = \partial_\mu v^\mu + \frac{1}{\sqrt{-g}} v^\mu \partial_\mu \sqrt{-g}$.)

Finally:

$$\delta S_{grav.} = \frac{M^2}{2} \int d^4x \sqrt{-g} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu}.$$

We now turn to the matter action and write:

$$\delta S_{matter} = \int d^4x \left( \frac{\delta}{\delta g^{\mu\nu}(x)} S_{matter} \right) . \, \delta g^{\mu\nu}(x)$$

Defining $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$, (the <u>energy-momentum tensor</u>)

we have

$$\delta S_{matter} = \int d^4x \sqrt{-g} \left( -\frac{1}{2} T_{\mu\nu} \right) \delta g^{\mu\nu}.$$

Setting the coefficient of $\delta g^{\mu\nu}$ in the variation of the full action to zero, we find the Einstein equation:

$$\boxed{M^2 \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = T_{\mu\nu}}$$

Next, we will discuss in more detail the quantity $T_{\mu\nu}$ at the r.h. side of the Einstein eq., which characterizes the density of energy and 4-momentum of matter and which is the source for the gravitational field.

## <u>4.2 The energy-momentum tensor</u>

In general, the matter action does not involve derivatives of the metric. Thus

$$S_{matter}[\varphi, g_{\mu\nu}] = \int d^4x \sqrt{-g} \, \mathcal{L}(\varphi, \partial\varphi, g^{\mu\nu}).$$

$$\Rightarrow \quad -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} S_{matter} = -\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} \left( \sqrt{-g}\, \mathcal{L} \right).$$

- As the simplest possible example, consider $\mathcal{L} = -\Lambda$, i.e. a cosmol. constant or vacuum energy density (positive $\Lambda$ corresponds to positive potential energy). Since we already know that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}$, we have

$$\frac{\partial}{\partial g^{\mu\nu}} \sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}$$

and $$T_{\mu\nu}^{\Lambda} = -\frac{2}{\sqrt{-g}} \frac{\partial}{\partial g^{\mu\nu}} \left( \sqrt{-g}\,(-\Lambda) \right) = -\Lambda\, g_{\mu\nu}.$$

- Thus, choosing locally $g_{\mu\nu} = \eta_{\mu\nu}$, $T_{00}{}^{\Lambda} = \Lambda$ is the energy density (this will be the case also for more general actions).

- The next simplest example is a point particle, for which we find:

$$T_{\mu\nu}(x) = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}(x)} \left( m \int d\tau \sqrt{-\dot{y}_\mu \dot{y}_\nu g^{\mu\nu}} \right)$$

$$= \frac{2m}{\sqrt{-g}} \int d\tau \frac{1}{2\sqrt{-\dot{y}^2}}\, \dot{y}_\mu \dot{y}_\nu\, \delta^4(x - y(\tau)))$$

$$\underset{\substack{\uparrow \\ \text{using a } \tau \text{ such} \\ \text{that } \dot{y}^2 = -1}}{=} \frac{m}{\sqrt{-g}} \int d\tau\, u_\mu u_\nu\, \delta^4(x - y(\tau)) \quad \text{with } u_\mu = \dot{y}_\mu \text{ the 4-velocity}.$$

- In particular, in special relativity or in a small patch of the manifold with a coordinate choice such that $g_{\mu\nu} = \eta_{\mu\nu}$:

$$\boxed{\begin{gathered} T_{\mu\nu}(x) = m \int d\tau\, \delta^4(x - y(\tau))\, u_\mu(\tau) u_\nu(\tau) \\ (\tau - \text{eigentime}) \end{gathered}}$$

- $\int d\tau\, \delta(x^0 - y^0(\tau)) = \left(\frac{dy^0(\tau)}{d\tau}\right)^{-1} = \gamma^{-1}$ $\quad$ ($\gamma = 1/\sqrt{1-\bar{v}^2}$ is the time-dilatation factor)

$$\Rightarrow \quad T_{\mu\nu}(x) = m\gamma^{-1}\delta^3(\bar{x} - \bar{y}(\tau))\, u_\mu(\tau) u_\nu(\tau)$$

- Let us now consider small volume $V$ of an ideal fluid (or gas) in its rest frame. We model this by a summation over single-particle $T_{\mu\nu}$'s as above, each averaged over $V$:

$$T_{\mu\nu} = \frac{1}{V}\sum_A m_A \gamma_A^{-1} u_\mu^A u_\nu^A .$$ (A labels the particles inside V)

- By the rest frame assumption,

$$T_{0i} = \frac{1}{V}\sum_A m_A \gamma_A^{-1} u_0^A u_i^A = 0$$ (no preferred sign of $u_i^A$).

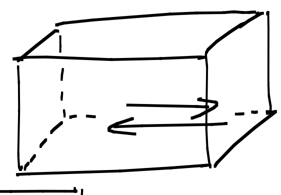
- Similarly $T_{i0} = 0$ & $T_{ij} = 0$ for $i \neq j$.

- $T_{00} = \frac{1}{V}\sum_A m_A \gamma_A^{-1}(u_0^A)^2 = \frac{1}{V}\sum_A m_A \gamma_A^{-1}\gamma_A^2 = \frac{1}{V}\sum_A E_A = \rho$

- $T_{11} = \frac{1}{V}\sum_A m_A \gamma_A^{-1}(u_1^A)^2$

By considering a particle bouncing back-and-forth in the $x^1$-direction in a box, one can show that $T_{11}$ is precisely its contribution to the pressure in $x^1$-direction,

i.e. $T_{11} = \sum_A p_{(1)}^A$. (Problem)



- Thus, by isotropy: $\boxed{T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)}$

<u>Problem:</u> Show that in the highly-relativistic case (e.g. photon gas), $p = \varrho/3$.

- The above form was "derived" for an ideal fluid in its rest frame. Characterizing the motion of the fluid by a vector field $u_\mu$, we can write the general expression:

$$\boxed{T_{\mu\nu} = (\varrho + p)\, u_\mu u_\nu + p\, g_{\mu\nu}}$$

- To see this, just focus on some point $x$ and choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $u^\mu = (1,0,0,0)$ (local Lorentz frame & rest frame of fluid). Then $T_{\mu\nu}$ takes the rest frame form derived earlier. This proves the correctness of the general formula since the r.h. side is obviously a tensor.

<u>Comment 1:</u>

- A different concept of energy-momentum tensor exists in special-relativistic classical field theory:

  - translation-symm. in $x^\mu$ imply conserved quantities (Noether charges) $P^\mu$.

  - these 4-momenta of the field-configuration follow from conserved currents:

$$\text{general:} \quad Q = \int d^3\bar{x}\, j^0 \quad \text{with } \partial_\mu j^\mu = 0$$

$$\text{here:} \quad P^\nu = \int d^3\bar{x}\, T^{0\nu} \quad \text{with } \partial_\mu T^{\mu\nu} = 0$$

- Our definition of $T^{\mu\nu}$ (by variation w.r.t. $g^{\mu\nu}$) can be can be compared with the Noether-definition (i.e. the "canonical" energy-momentum tensor) if the limit $g_{\mu\nu} \to \eta_{\mu\nu}$ is taken after variation:

$$T_{GR}^{\mu\nu} = T_{canon.}^{\mu\nu} + \partial_\rho \Sigma^{\mu\nu\rho}$$

always symm. in $\mu,\nu$

in general not symm.

some tensor antisymm. in $\mu,\rho$ (so that $\partial_\mu T^{\mu\nu}_{canon.} = 0$ $\Updownarrow$ $\partial_\mu T^{\mu\nu}_{GR} = 0$.)

- Problem: Check this for a real scalar field (where $\Sigma = 0$) and (if you are really interested), for electrodynamics (where $\Sigma \neq 0$).

Comment 2:

Since $T^{\mu\nu}$ is the $\mu$-component of the current of $P^\nu$-density, we can say:

"$T^{\mu\nu}$ is the flux of 4-momentum $P^\nu$ across a surface of constant $x^\mu$."

(Easy problem: Derive the form $T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$ directly from this definition.)

## 4.3 Newtonian Limit

- Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($h_{\mu\nu}$ small) & $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$; ("Linearization") $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$

$$\Rightarrow \Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \cdots\cdots)$$

$$\Rightarrow R_{\mu\nu} = -\partial_\mu \Gamma_{\rho\nu}{}^{\rho} + \partial_\rho \Gamma_{\mu\nu}{}^{\rho}$$ (terms with 2 factors of $\Gamma$ are higher order)

<u>Problem:</u> Derive from this the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ in linear oder in $h_{\mu\nu}$.

--- ⇒ Einstein eq.: [with $\underline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ ; $h = h_\mu{}^\mu$ ]

$$G_{\mu\nu} = -\frac{1}{2}\partial^\rho\partial_\rho \underline{h}_{\mu\nu} + \partial^\rho \partial_{(\mu} \underline{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \underline{h}_{\rho\sigma} = \frac{1}{M^2} T_{\mu\nu}$$

- Let us now consider the possibility of "small" diffeomorphisms: $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x)$

$$\Rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) =$$

$$= (\delta_\mu^\rho - \partial_\mu \xi^\rho)(\delta_\nu^\sigma - \partial_\nu \xi^\sigma)\, g_{\rho\sigma}(x)$$

$$\Rightarrow \cancel{\eta_{\mu\nu}} + \underline{h'_{\mu\nu}} = \cancel{\eta_{\mu\nu}} + \underline{h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu}$$

This is the gauge freedom of linearized gravity (analogous to $A'_\mu = A_\mu - \partial_\mu \chi$ of electrodynamics).

(<u>Note:</u> We do not need to correct for $x \neq x'$ in the arguments on the l.h. & r.h. sides of these eqs. since $\eta_{\mu\nu}$ is const. and in $h_{\mu\nu}$ this effect is higher order.)

- $\partial^\mu h'_{\mu\nu} = \partial^\mu h_{\mu\nu} - \partial^\mu\partial_\mu \xi_\nu - \partial^\mu \partial_\nu \xi_\mu$

$\partial^\mu \underline{h}'_{\mu\nu} = \partial^\mu \underline{h}_{\mu\nu} - \partial^\mu\partial_\mu \xi_\nu - \partial^\mu\partial_\nu \xi_\mu + \frac{1}{2}\partial^\mu(\eta_{\mu\nu}\cdot 2\partial^\rho \xi_\rho)$

$$= \partial^\mu \underline{h}_{\mu\nu} - \partial^\mu \partial_\mu \xi_\nu \underset{\Uparrow}{=} 0$$

We can always find a $\xi_\nu$ such that this is true (cf. existence of potential in electrostatics).

- We can choose to work in the gauge $\boxed{\partial^\mu \underline{h}_{\mu\nu} = 0}$

$\Rightarrow$ The Einstein eq. simplifies: $\partial^\mu \partial_\mu \underline{h}_{\varrho\sigma} = -\frac{2}{M^2} T_{\varrho\sigma}$

- The Newtonian limit requires that, in some coord. system, all sources move slowly ($v \ll 1$). In such a situation pressure & stresses (off-diag. terms of $T_{\mu\nu}$) are usually small. (See e.g. our ideal fluid example above.)

- Thus, we assume $\underline{T_{\mu\nu}(x) = \varrho(x)\, u_\mu u_\nu}$ (with $u_\mu = (1,0,0,0)$ in this system.)

- Since the sources move slowly, we can also assume that time derivatives of $\underline{h}_{\mu\nu}$ are small.

- Einstein eqs.: $\Delta \underline{h}_{ij} = 0$ ; $\Delta \underline{h}_{i0} = 0$ $(\Delta = \partial^i \partial_i$ ; $i = 1..3)$

$$\Delta \underline{h}_{00} = -\frac{2}{M^2} \varrho$$

($\underline{h}_{ij}$ & $\underline{h}_{i0}$ vanish everywhere)

- Thus, $\underline{h}_{00}$ plays (roughly) the role of the gravitational potential of Newtonian gravity.

- Particle motion: $\ddot{x}^\mu + \Gamma_{\varrho\sigma}{}^\mu \underbrace{\dot{x}^\varrho \dot{x}^\sigma}_{} = 0$
(geodesics!)

set $\dot{x}^\varrho \simeq (1, \vec{0})$ in this term

- $\Gamma_{00}{}^{i} = -\frac{1}{2}\eta^{ij}\partial_j h_{00} = -\frac{1}{2}\partial^i h_{00}$ (neglecting time-deriv. of $h_{\mu\nu}$)

- Need to relate $h_{00}$ & $\underline{h}_{00}$:

$$\underline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad \Rightarrow \underline{h} = h - 2h \Rightarrow h = -\underline{h}$$

$$\Rightarrow h_{\mu\nu} = \underline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\underline{h}$$

$$h_{00} = \underline{h}_{00} - \frac{1}{2}\eta_{00}(-\underline{h}_{00}) = \frac{1}{2}\underline{h}_{00}$$

- $\Rightarrow \Gamma_{00}{}^{i} = -\partial^i \frac{1}{4}\underline{h}_{00} \quad \Rightarrow \ddot{x}^i = \partial^i\left(\frac{1}{4}\underline{h}_{00}\right) \equiv -\partial^i\phi$

grav. potential of Newt. gravity

$$\Rightarrow \phi = -\frac{1}{4}\underline{h}_{00}$$

$$\Delta\phi = \frac{1}{2M^2}\rho \quad \text{— comare with —} \quad \Delta\phi = \underbrace{4\pi}G_N\rho$$

$$\Rightarrow M^2 = \frac{1}{8\pi G_N} = \frac{M_P^2}{8\pi}$$

These $4\pi$ come, of course, from the desire to have no $4\pi$ in the force law.

<u>Finally</u>: $h_{\mu\nu} = \underline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\underline{h} = \underline{h}_{00}\left(u_\mu u_\nu + \frac{1}{2}\eta_{\mu\nu}\right)$

$$= -(4u_\mu u_\nu + 2\eta_{\mu\nu})\phi$$

Thus, we have fully derived Newton's theory in terms of $\phi$ and $\ddot{x}^i$, and we have found the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in terms of $\phi$.

## 4.4 Electromagnetic radiation in Coulomb gauge

(as a preparation for gravity waves)

- We first use gauge freedom, $A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$ to enforce the Lorentz gauge, $\partial_\mu A^\mu = 0$.

  (Given some $A_\mu$, this requires finding a fct. $\chi$ such that
  $$0 = \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi .$$
  Such a $\chi$ always exists, since the above can be read as the defining equation of the electrostatic potential for a given charge distribution.)

- Next, given an $A_\mu$ with $\partial_\mu A^\mu = 0$ and <u>no charges</u>, we want to find a further gauge tf. $A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$, such that $A_0 = 0$ (Coulomb gauge).

- If we require that our new $\chi$ satisfies $\partial_\mu \partial^\mu \chi = 0$, the Lorentz gauge will not be destroyed:
  $$\partial_\mu A'^\mu = \partial_\mu (A^\mu + \partial^\mu \chi) = \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi = 0 .$$

  (i.e., we are making use of the <u>residual gauge freedom</u>)

- Thus, we need a $\chi$ such that $\partial_\mu \partial^\mu \chi = 0$ & $A_0 + \partial_0 \chi = 0$.

- $\partial^2 \chi = 0$ is a 2nd order diff. eq. in $t$. It will always have a solution throughout all space if we manage to supply boundary conditions for $\chi$ & $\partial_t \chi$ on some <u>initial surface</u> $t_0$ (consistent with our requirement $A_0 + \partial_0 \chi = 0$)

- To do so, define $f = A_0 + \partial_0 \chi$ and demand

$$f = 0 \quad \& \quad \partial_t f = 0 \quad \text{at } t_0 :$$

1) $\partial_0 \chi = - A_0$

2) $\partial_0 A_0 + \partial_0^2 \chi = \partial_i A_i + \partial_i \partial_i \chi = 0$ or $\nabla^2 \chi = - \nabla \bar{A}$.

- 1) & 2) unambiguously defines $\chi$ and $\partial_t \chi$ at $t_0$.
  $\partial^2 \chi = 0$ then defines $\chi$ for all $t > t_0$.

- We now only need to check that our $\chi$ indeed realizes the Coulomb gauge:
  We already know $f = 0$ & $\partial_t f = 0$ at $t_0$. Furthermore,

$$\partial^2 f = \partial^2 (A_0 + \partial_0 \chi) = \partial_\mu \partial^\mu A_0 + \partial_0 \partial^2 \chi = 0 + 0 = 0$$

from the Maxwell eq. from $\partial^2 \chi = 0$.

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu = 0$$

$$\& \ \partial_\mu A^\mu = 0$$

This defines $f$ uniquely and $f = 0$ is clearly a solution.

## 4.5 Gravity waves

- We now repeat the above procedure for linearized gravity.

- We assume that we have already found $h_{\mu\nu}$ with $\partial^\mu \underline{h}_{\mu\nu} = 0$ ("Lorentz gauge").

- In addition, we want $h = 0$ and $h_{0i} = 0$. We are only allowed to use $\xi_\mu$ with $\partial^2 \xi_\mu = 0$ since we want to maintain Lorentz gauge:

$$\partial^\mu \underline{h}'_{\mu\nu} = \partial^\mu \left( h'_{\mu\nu} - \tfrac{1}{2} \eta_{\mu\nu} h \right) = \partial^\mu \Big( h_{\mu\nu} - \partial_\mu \chi_\nu - \partial_\nu \chi_\mu - \tfrac{1}{2} \eta_{\mu\nu} \left( h_\mu{}^\mu - 2 \partial_\mu \chi^\mu \right) \Big) = 0.$$

- We again choose an initial surface $t_0$ and, given appropriate boundary conditions at $t_0$, define $\xi_\mu$ for all $t > t_0$ by $\partial^2 \xi_\mu = 0$.
- The boundary conditions follow, as before from $h' = 0$ & $h'_{0i} = 0$ and their first time derivatives at $t_0$:

1) $0 = h' = h - 2\partial_\mu \xi^\mu \quad \Rightarrow \quad 0 = h - 2(\partial_t \xi^0 + \bar{\nabla}\bar{\xi})$

2) $0 = \partial_t h' = \partial_t h - 2\partial_t \partial_\mu \xi^\mu = \partial_t h - 2(\partial_t^2 \xi^0 + \partial_t \bar{\nabla}\bar{\xi})$

$$\Rightarrow 0 = \partial_t h - 2(-\bar{\nabla}^2 \xi^0 + \bar{\nabla}(\partial_t \bar{\xi}))$$

3) $0 = h'_{0i} = h_{0i} - 2\partial_t \xi_i - 2\partial_i \xi_0$

4) $0 = \partial_t h'_{0i} = \partial_t h_{0i} - \partial_t(\partial_t \xi_i + \partial_i \xi_0)$

$$\Rightarrow 0 = \partial_t h_{0i} - \bar{\nabla}^2 \xi_i - \partial_i(\partial_t \xi_0)$$

- These 8 equations unambiguously determine $\xi_0$, $\partial_t \xi_0$, $\bar{\xi}$ and $\partial_t \bar{\xi}$ at $t_0$. (e.g., solve 1) & 3) for $\partial_t \xi^0$ & $\partial_t \xi^i$ and insert in 2) & 4) to get 2nd order diff. eqs. on the $t_0$ surface for $\xi_0$ & $\bar{\xi}$)

To see how this works, consider e.g. Eq. 1) and solve it for $\partial_t \xi^0$:

$$\partial_t \xi^0 = -\bar{\nabla}\bar{\xi} + \frac{1}{2} h .$$

Inserting this into Eq. 4), we find

$$0 = \partial_t h_{0i} - \bar{\nabla}^2 \xi_i - \partial_i \bar{\nabla}\bar{\xi} - \frac{1}{2}\partial_i h$$

$$\underbrace{(\bar{\nabla}^2 \delta_{ij} + \partial_i \partial_j)}\, \xi_j = \partial_t h_{0j} - \frac{1}{2}\partial_j h$$

This is an invertible operator, as one can easily see in

Fourier space: $\bar{\nabla}^2 \delta_{ij} + \partial_i \partial_j \longrightarrow \bar{k}^2 \delta_{ij} + k_i k_j$

ansatz: $(\bar{k}^2 \delta_{ij} + k_i k_j)(a \delta_{jk} + b k_j k_k) = \delta_{ik}$

$$\delta_{ik} a \bar{k}^2 + (a + 2b\bar{k}^2) k_i k_k = \delta_{ik}$$

$\Rightarrow a = 1/\bar{k}^2 \;\; ; \;\; b = -1/2(\bar{k}^2)^2$

Thus, we have determined $\bar{\xi}$ as well as $\partial_t \xi_0$.
We can similarly determine $\xi_0$ and $\partial_t \bar{\xi}$.

- Now define $\xi_\mu$ as the solution of $\partial^2 \xi_\mu = 0$ with these boundary conditions.

- In complete analogy to electrodynamics, we now use the source-free Einstein eqs. to show that $\partial^2 (h - 2\partial_\mu \xi^\mu)$ and $\partial^2 (h_{0i} - (\partial_0 \xi_i + \partial_i \xi_0))$ vanish everywhere.

In more detail: $\partial^2 (h - 2\partial_\mu \xi^\mu) = \partial^2 h = 0$

This follows from the trace of the Einstein eq. $\partial^2 \underline{h}_{\mu\nu} = -\frac{2}{M^2} T_{\mu\nu}$ for $T_{\mu\nu} = 0$.

(and similarly for the other condition)

Given the initial conditions on these quantities, we have indeed achieved the desired gravitational <u>radiation gauge</u>.

- We automatically get $h_{00} = 0$:
  - $h = 0 \Rightarrow \underline{h}_{\mu\nu} = h_{\mu\nu} \Rightarrow$ the Einstein eqs. read
    $$\partial^2 h_{\mu\nu} = -\frac{2}{M^2} T_{\mu\nu} = 0$$
    (no sources)
  - Furthermore, $0 = \partial^\mu h_{0\mu} = -\partial_t h_{00} + \partial_i \underbrace{h_{0i}}_{=0}$
    $$\Rightarrow \partial_t h_{00} = 0$$
  - This can only be solved by $h_{00} = $ const.
    $h_{00} = 0$ can now always be achieved by a gauge trf. (with $\xi_0 \sim t$ and, e.g. $\xi_i \sim x^i$) respecting all previous constraints.
- Thus, we have achieved: $\partial^\mu h_{\mu\nu} = 0$; $h_{00} = 0$; $h = 0$; $h_{0i} = 0$
- We make the ansatz $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ikx} = \varepsilon_{\mu\nu} e^{i(-k^0 t + \bar{k}\bar{x})}$

  (plane wave)

  $\partial^2 h_{\mu\nu} = 0$ requires $k^2 = 0$ (as for the photon).
- Furthermore, we have $\underbrace{k^\mu \varepsilon_{\mu\nu} = 0}$; $\varepsilon_{00} = 0$; $\varepsilon_\mu{}^\mu = 0$; $\varepsilon_{0i} = 0$

  of these 4 constraints, the constraint $k^\mu \varepsilon_{\mu 0}$ is trivial since $\varepsilon_{00} = \varepsilon_{0i} = 0$

  $\Downarrow$

  3 constraints

  $\underbrace{\hspace{6cm}}$

  8 constraints in total
- Of the $(4 \cdot 4 - 4)/2 + 4 = 10$ elements of $\varepsilon_{\mu\nu}$ only

two are independent $\Rightarrow$ <u>2 polarizations of the graviton</u>.

## <u>4.6 Normal coordinates and the equivalence principle</u>

- A <u>normal coordinate system</u> $x$ at a point $p \in M$ is defined by the property $g_{\mu\nu}(p) = \eta_{\mu\nu}$.
- For <u>Riemannian normal coordinates</u> (sometimes also just called "normal coordinates") we also require

$$\partial_\mu g_{\nu\rho}(p) = 0.$$

- To construct Riemannian normal coordinates, we first choose a basis of $T_p$ such that $g(\tilde{e}_{(\mu)}, \tilde{e}_{(\nu)}) = \eta_{\mu\nu}$. For each $v \in T_p$, we then define a geodesic $x_v(\tau)$ ($x$ are <u>some</u> coordinates) with $x(0) = x(p)$ and $\dot{x}_v^\mu(0) = v^\mu$:
$$\ddot{x}_v^\mu + \Gamma_{\nu\rho}{}^\mu \dot{x}_v^\nu \dot{x}_v^\rho = 0.$$
(This always has a unique solution by the theory of ordinary diff. equations.)
- We now define a map from $T_p \simeq \mathbb{R}^n$ to the manifold by
$$T_p \longrightarrow M$$
$$v \longmapsto \varphi^{-1}(x_v(1))$$
(i.e. by going a distance 1 in the direction of $v$ along the uniquely defined geodesic).
- This map, which is also known as the "exponential map", is a diffeomorphism in a neighbourhood of $p$.

- If we now parametrize $T_p$ by $v = v^\mu \tilde{e}_{(\mu)}$, the we have found our <u>Riem. normal coordinate system</u> $v^\mu$.

- <u>Check:</u> 1) Form of the metric:

$$g_{\mu\nu}(0) = g\left(\frac{\partial}{\partial v^\mu}, \frac{\partial}{\partial v^\nu}\right) \underset{\uparrow}{=} g(\tilde{e}_{(\mu)}, \tilde{e}_{(\nu)}) = \eta_{\mu\nu}$$

Since $\frac{d}{d\tau}\left(\tilde{e}_{(\mu)}\text{-geodesic}\right)^\nu = \frac{d}{d\tau}\left(\delta^\nu{}_\mu \cdot \tau\right) =$

$$= \delta^\nu_\mu = \left(\frac{\partial}{\partial v^\mu}\right)^\nu$$

2) First derivatives:

Apply the geodesic equation to the geodesic $v^\mu\tau$ defined by $v \in T_p$:

$$(v^\mu\tau)^{\cdot\cdot} + \Gamma_{\nu\rho}{}^\mu (v^\nu\tau)^\cdot (v^\rho\tau)^\cdot = 0$$

$$\Gamma_{\nu\rho}{}^\mu v^\nu v^\rho = 0 \quad \text{for all } v.$$

Since $\Gamma_{\nu\rho}{}^\mu$ is symm. in $\nu$ & $\rho$, this implies $\Gamma_{\nu\rho}{}^\mu = 0$. Furthermore: $0 = D_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho}$ $\square$.

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<u>Comment:</u> Unrelated to our main line of thought in this chapter, there also exists the concept of <u>Gaussian normal coordinates</u>: Given a hypersurface $H$ of an $n$-dim. Riemannian manifold $M$, parametrized by $(x^1, \dots, x^{n-1})$, we can define the (normalized) <u>normal vector</u> at every point of this hypersurface and parametrize $M$ near $H$ by moving a distance $t$ along these geodesics from any point on $H$. $M$ is then

locally described by coordinates $(x^1, \cdots, x^{n-1}, t)$. 
[On a Lorentz manifold H must be "non-null", i.e. none of its normal vectors should be light-like.]

- We now formulate the <u>Einstein equivalence principle</u>:

  ‖ In small regions of space-time, the (non-gravitational) laws of physics reduce to those of special relativity. ‖

  (i.e. gravity can not be detected by sufficiently local measurements.)

- Given our knowledge about Riem. normal coordinates, this means that we can always go, at some point $p \in M$, to Riem. normal coordinates and describe physics exactly as in special relativity:

  - particle motion: $\ddot{x}^\mu = 0$
  - electrodynamics: $\partial_\mu F^{\mu\nu} = j^\nu$
  - energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0$

- Given our requirement of general covariance, we then find that in any coordinate system:

  - particle motion: $\ddot{x}^\mu = \dot{x}^\varrho \frac{\partial}{\partial x^\varrho} \cdot \frac{d}{d\tau} x^\mu = 0$

    $$\Rightarrow \dot{x}^\varrho D_\varrho \dot{x}^\mu = \ddot{x}^\mu + \Gamma_{\varrho\nu}{}^\mu \dot{x}^\varrho \dot{x}^\mu = 0$$

  - electrodynamics: $D_\mu F^{\mu\nu} = j^\mu$

  - covariant conservation of energy-momentum tensor: $D_\mu T^{\mu\nu} = 0$.

- This can be used as the basis for defining the dynamics of matter in gravitational fields. (Although I have put more emphasis on the method of writing down "natural" diff.-invariant lagrangians.)

- In the "equivalence principle approach", one basically just writes down special relativistic equations and then replaces $\partial_\mu \to D_\mu$. (Some authors denote these types of differentiation by commas and semi-colons:

$$,_\mu \to ;_\mu \,,$$

$$\text{e.g. } F^{\mu\nu}{}_{,\mu} = j^\nu \quad \to \quad F^{\mu\nu}{}_{;\mu} = j^\nu .)$$

- The Einstein equivalence principle can, e.g., be violated if the matter lagrangian contains $R_{\mu\nu\rho}{}^\sigma$ (together with matter fields). Clearly, the EOMs then do not follow from the "comma-to-semi-colon" rule.

- There also exists the <u>strong equivalence principle</u>, saying that <u>all</u> (including gravitational) laws of physics reduce locally to special relativity. [E.g., a body whose mass depends on gravitational binding energy should fall in the same way as other bodies. It can be violated by higher-curvature-terms in the gravitational action.]

- Furthermore, the <u>weak equivalence principle</u> states that all bodies fall in the same way. This is a "trivial" consequence of the motion on geodesics (i.e. the mass or any other feature of the body simply does not enter). It can be violated, e.g., if the spin of the body couples to gravity in some non-standard way (see later).

- All equivalence principles (in particular the "most experimental" weak one) can clearly be violated if massless fields are, in addition to $g_{\mu\nu}$, are part of the gravitational action ("scalar-tensor-theories").