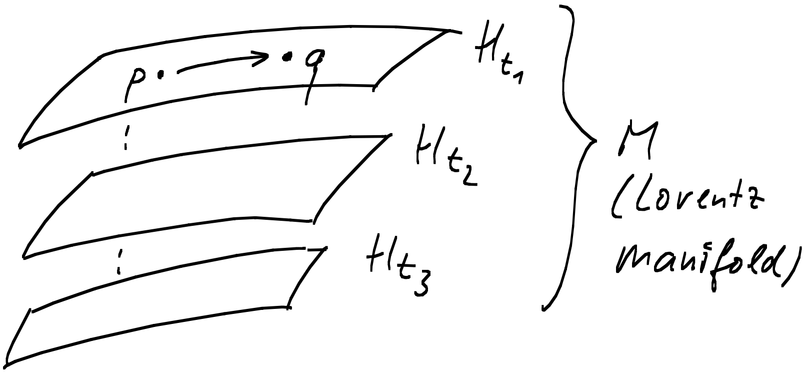
5 Cosmology

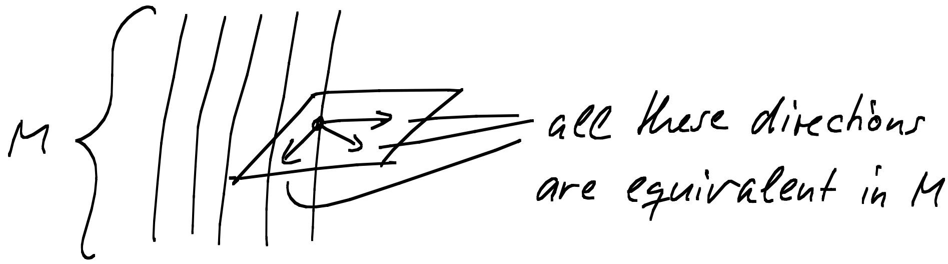
5.1 Homogeneity & Isotropy

- A spacetime is <u>spatially</u> homogeneous if $\exists$ a 1-parameter family of spacelike hypersurfaces H_t $(t \in \mathbb{R})$, each of which is <u>homogeneous</u>.

- A manifold is homogeneous if for any $p, q \in M$ $\exists$ an isometry (a diffeomorphism $f: M \to M$ leaving the metric invariant, i.e. not changing distances) with $f(p) = q$.



- We also require <u>spatial</u> isotropy, i.e. the existence of a congruence of "observers" (time-like curves) such that at any point of any of these curves the tangent subspace orthogonal to that curve has no distinguished directions.



(involving "observers" in the definition of isotropy is unavoidable since, e.g., in a non-relativistic gas only <u>one</u> of all possible observers at a given point can see an isotropic environment)

- Homogeneity & isotropy taken together imply that the H_t are orthogonal to the "observer" curves (otherwise we could distinguish directions orthogonal to an observer by their angle relative to the appropriate H_t).

- $g_{\mu\nu}$ on M induces a metric $\tilde{g}_{ij}$ on H_t (two nearby points in H_t can be viewed as points in M and their distance can be measured with $g_{\mu\nu}$; this defines their distance in H_t measured with $\tilde{g}_{ij}$ and thereby $\tilde{g}_{ij}$ itself).

- The curvature $\tilde{R}_{ij}{}^{kl}$ of H_t can be viewed as a liner map on $(T_p^* \otimes T_p^*)_{\text{antisymmetrized}}$ (i.e. on 2-forms).

  In any basis of 2-forms, this map is given by a symm. matrix (cf. properties of $R_{\mu\nu\rho\sigma}$). Thus, if it was not $\sim Id$, we could pick out distinguished 2-forms (eigenvectors with specific eigenvalues) and hence distinguished vectors (e.g. as $v^i = \varepsilon^{ijk}\omega_{jk}$ for a distinguished $\omega \in (T_p^* \otimes T_p^*)_A$).

- Thus: $$\tilde{R}_{ij}{}^{kl} = \kappa\,(\delta_i{}^k \delta_j{}^l - \delta_i{}^l \delta_j{}^k)$$

  (with κ constant by homogeneity)

  (Spaces with such a curvature tensor are called "maximally symmetric" or "spaces of constant curvature.)

- It is a non-trivial theorem (although it is intuitively clear if we believe that $R_{\mu\nu\rho\sigma}$ contains all geometric information) that spaces with const. curvature are (locally) completely specified by dimension, metric signature and κ.

- Thus, we only need to find examples for any value of κ to be able to describe the most general (hom. & isotr.) cosmology.

① $K = 0$: This is clearly achieved by flat 3d space, i.e.

$$ds^2 = dx^2 + dy^2 + dz^2$$

② $K > 0$: This is realized by 3-spheres, which can be described as submanifolds of $\mathbb{R}^4$:

$$w^2 + x^2 + y^2 + z^2 = R^2 .$$

Parameterization by spherical coordinates (for $R = 1$)

$$\left.\begin{aligned} w &= \cos\psi \\ x &= \sin\psi \cos\theta \\ y &= \sin\psi \sin\theta \cos\varphi \\ z &= \sin\psi \sin\theta \sin\varphi \end{aligned}\right\} \text{ from this, } w^2+x^2+y^2+z^2 = 1 \text{ is immediately obvious}$$

To get the metric, consider

$$\begin{aligned} dw &= -\sin\psi \, d\psi \\ dx &= \cos\psi \cos\theta \, d\psi - \sin\psi \sin\theta \, d\theta \\ dy &= \cos\psi \sin\theta \cos\varphi \, d\psi + \sin\psi \cos\theta \cos\varphi \, d\theta - \sin\psi \sin\theta \sin\varphi \, d\varphi \\ dz &= \cos\psi \sin\theta \sin\varphi \, d\psi + \sin\psi \cos\theta \sin\varphi \, d\theta + \sin\psi \sin\theta \cos\varphi \, d\varphi \end{aligned}$$

It is easily seen that in $ds^2 = dw^2 + \dots + dz^2$ all "cross-terms" drop out (this is true systematically in the above, standard way of parameterizing spheres).

One then finds:

$$\begin{aligned} ds^2 &= d\psi^2 + \sin^2\psi \, d\theta^2 + \sin^2\psi \sin^2\theta \, d\varphi^2 \\ &= d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \end{aligned}$$

The maximal symmetry is also obvious: $SO(4)$ leaves the euclidean metric on $\mathbb{R}^4$ invariant and maps our submanifold S^3 onto itself. Hence each of its elements represents an isometry (in the induced metric) of S^3. Clearly, any point of S^3 can by mapped by SO_4 to any other point. Furthermore, fixing a point ($\hat{=}$ fixing a vector $\in \mathbb{R}^4$), we can still rotate about this point using an $SO(3)$ subgroup.

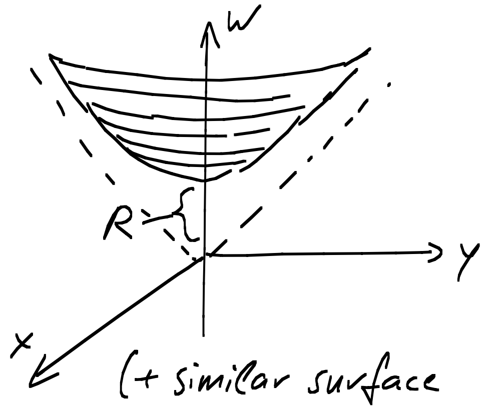
(e.g. fix $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \in \mathbb{R}^4$; $\left(\begin{array}{c|c} 1 & \\ \hline & SO(3) \end{array}\right) \subset SO(4)$ can still be used)

This "surviving" $SO(3)$-symmetry is sufficient to map any tangent vector at the fixed point into any other tangent vector. $\Rightarrow$ S^3 is homogeneous & isotropic.

③ $K < 0$: This can be realized by a hyperboloid, which can be described by a submanifold of $\mathbb{R}^4$ with Lorentz metric (i.e. Minkowski space):

$$-w^2 + x^2 + y^2 + z^2 = -R^2$$

To get some intuition, we draw this space one less dimension:



(+ similar surface at $w < 0$)

The parametrization is exactly as for the sphere, but with

$$\cos\psi \to \cosh\psi$$
$$\sin\psi \to \sin\psi,$$

giving the metric $ds^2 = d\psi^2 + (\sinh\psi)^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$

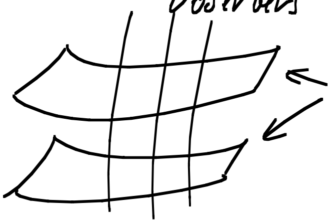
- In analogy to the case of the sphere, we use the metric on our submanifold which is induced by the Minkowski metric on $\mathbb{R}^4$. This metric (and of course the hyperboloid) is invariant under $SO(1,3)$. As before, $SO(1,3)$ is sufficient to map any point on the (upper part of) our hyperboloid to any other point. Fixing a point (take e.g. $(1,0,0,0)$), we can still use

$$\left(\begin{array}{c|c} 1 & \\ \hline & SO(3) \end{array}\right) \subset SO(1,3)$$

to perform further rotations. This establishes isotropy at this point (and hence everywhere).

[For the calculation of the curvature see problem 2 on sheet 4. This easily generalizes to S^3. In the hyperboloid case we get an extra "$-$" from $\sin \to \sinh$. $\Rightarrow \mathcal{R} = \pm R$. This fixes κ.]

- We now return to our foliated cosmological spacetime:



$\Big\{$ flat $\mathbb{R}^3$ — "open"
or S^3 — "closed" universe (compact space)
or hyperboloids — "open"

<u>Note:</u> It is possible to construct compact universes with flat or neg. curved metric. The obvious example is the torus T^3.

- Using the orthogonality of observer worldlines and homog. spatial hypersurfaces, we see that the full metric can be written as $ds^2 = \underbrace{-d\tau^2}_{\text{observer eigentime}} + \underbrace{a^2(\tau)}_{\text{"scale factor"}} \underbrace{g_{ij}\, dx^i dx^j}$

dimensionless ($R=1$) metric of flat $\mathbb{R}^3$, S^3 or hyperboloid

5.2 Einstein Equations

- We assume that the matter in the universe is a perfect fluid:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu)$$

clearly, this must also be the unit tangent vector of our "observer worldlines".

- Einstein eqs.: $M^2 G_{\mu\nu} = T_{\mu\nu}$ $\quad (G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu})$

- All information can be extracted by contracting with all combinations of u^μ and $e^\mu_{(1)} \cdots e^\mu_{(3)}$ (unit tangent vectors of hypersurface).

- Clearly $u^\mu T_{\mu\nu} \sim u_\nu$ and $u^\mu G_{\mu\nu} \sim u_\nu$ (or isotropy would be violated. Hence, all "mixed" contractions with u^μ and $e_{(i)}{}^\mu$ are identically zero. Furthermore, $e^\mu_{(1)} \cdots e^\mu_{(3)}$ are equivalent.
$\Rightarrow$ only 2 independent equations:

$$M^2 G_{\tau\tau} = \rho$$
$$M^2 G_{\mu\nu} e^\mu e^\nu = p \quad \text{(any of the } e^\mu\text{'s)}$$

- Straightforward calculation gives $\quad (\dot{a} \equiv da/d\tau)$

$$G_{\tau\tau} = 3(\dot{a}^2 + k)/a^2 \quad ; \quad G_{\mu\nu} e^\mu e^\nu = -2\frac{\ddot{a}}{a} - (\dot{a}^2 + k)/a^2$$

k: $+1, 0, -1$ for S^3, $\mathbb{R}^3$, hyperboloid

(<u>Problem</u>: Check <u>at least</u> the flat case)

$\Rightarrow$ Einstein eqs.: $\quad (1)\ \ 3\dot{a}^2/a^2 = \rho/M^2 - 3k/a^2$

$\quad (2)\ -6\ddot{a}/a = (\rho + 3p)/M^2$

• Consider $\left((1)\cdot a^2\right)^{\bullet}$: $\quad 6\dot{a}\ddot{a} = 2(\rho/M^2)a\dot{a} + (\dot{\rho}/M^2)a^2$

or $6\ddot{a}/a = 2(\rho/M^2)\,\dot{a}/a + \dot{\rho}/M^2$

• Inserting this in (2) we find: $\quad \dot{\rho} + 3(\rho+p)\dfrac{\dot{a}}{a} = 0$

[This could also have been derived simply from $D_\mu T^{\mu\nu} = 0$ (continuity equation). Thus, we could have avoided deriving (2) altogether, working simply with (1) + continuity equation.]

• Now, defining $H \equiv \dot{a}/a$ ("Hubble parameter"), we have the complete "Friedmann-Robertson-Walker Cosmology" described by just two diff. equations:

$$\boxed{\begin{aligned} 3M^2H^2 &= \rho - 3M^2k/a^2 \\ \dot{\rho} + 3(\rho+p)H &= 0 \end{aligned}} \qquad \begin{aligned} &(\text{"FRW eq."}) \\ &(\text{continuity}) \end{aligned}$$

well-measured to be small (universe is "flat")

5.3 FRW - Cosmology

• <u>3 most important cases:</u> (with $k=0$, $M=1$)

① "dust" or dark matter or "just stars etc.": $\quad p = 0$

$$\dot{\rho} + 3H\rho = 0 \quad , \quad \rho = 3H^2 \Rightarrow \dot{\rho} = 6H\dot{H}$$

$$\Rightarrow \quad 2\dot{H} + 3H^2 = 0 \quad \Rightarrow \quad H = \frac{2}{3}\tau^{-1}$$ (integr. const. absorbed in $\tau \to \tau + c$)

$$\Rightarrow \quad (\ln a)^\cdot = \frac{2}{3}\tau^{-1} \quad \Rightarrow \quad a \sim \tau^{2/3}$$

(Note also that $\rho \sim \tau^{-2} \sim a^{-3}$, as it should be intuitively for "dust")

② radiation: $p = \rho/3$

$$\dot{\rho} + 4H\rho = 0 \,, \qquad \dot{\rho} = 6H\dot{H}$$

$$\Rightarrow \quad 2\dot{H} + 4H^2 = 0 \quad \Rightarrow \quad H = \frac{1}{2}\tau^{-1}$$

$$\Rightarrow \quad a \sim \tau^{1/2}$$

(Also: $\rho \sim a^{-4}$, as it should due to dilution of photons + their redshift)

③ cosm. const.: $p = -\rho$ (such that $T_{\mu\nu} \sim g_{\mu\nu}$)

$$\Rightarrow \dot{H} = 0 \,, \quad H = \text{const.} \,, \quad a \sim e^{H\tau}$$

(Also: $\rho = \text{const.}$)

<u>History of the universe:</u> (as far as we know or guess)

$t \downarrow$

③ Inflation

② radiation domination

① matter domination

③ cosm. const. domination

② radiation domination, ① matter domination: decelerated

③ Inflation, ③ cosm. const. domination: accelerated expansion