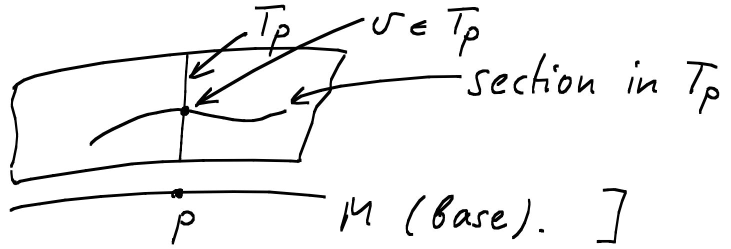
6 Differential Forms in General Relativity

6.1 Differential forms

- Recall that a vector field is a prescription assigning to every $p \in M$ an element of T_p.
  [More abstractly: all T_p's form the <u>tangent bundle</u> of M.
  A vector field is a <u>section</u> in the tangent bundle:



]

- Analogously, a 1-form is a prescription assigning to every $p \in M$ an element of $T_p{}^*$ (i.e., a section in the cotangent bundle).

- Any fct. f defines in a natural way a 1-form df by

$$df(v) \equiv v^\mu \partial_\mu f \quad (v \text{ is a vector field}).$$

- In particular, in a given parameterization any coordinate (viewed as a fct. on M) defines a 1-form dx^μ.

- Since $v = v^\mu \partial_\mu$, we see from the above action of 1-forms on vector fields that $\partial_\mu f$ are the components of df in the basis dual to $\{\partial_\mu\}$, i.e.

$$df = (\partial_\mu f)\, dx^\mu .$$

- We can call fcts. 0-forms and think of "d" as a map from 0-forms to 1-forms:

$$d: f \mapsto df .$$

- The natural extension to higher forms is as follows:

- A n-form ω_n (or just ω, if the degree is understood) is a prescription assigning to every $p \in M$ an element of $(\underbrace{T_p{}^* \otimes \cdots \otimes T_p{}^*}_{n \text{ factors}})_{\text{antisymmetrized}}$.

[$(T_p^* \otimes \cdots \otimes T_p{}^*)_{\text{antisymm.}} \equiv (T_p^* \otimes \cdots \otimes T_p^*)_A$ are linear functionals on $(T_p \otimes \cdots \otimes T_p)$ which are totally antisymm. in all their arguments, i.e

$$\omega_n(v_1, \ldots, v_k, \ldots, v_l, \ldots, v_n) = -\omega(v_1, \ldots, v_l, \ldots, v_k, \ldots, v_n).$$

We can also view elements of $(T_p^* \otimes \cdots \otimes T_p^*)_A = (T_p^{*\otimes n})_A$ as equivalence classes inside $T_p^{*\otimes n}$. The equivalence is defined by performing (as often as desired) the operation

$$a_1 \otimes \cdots \otimes a_k \otimes a_{k+1} \otimes \cdots \otimes a_n \longrightarrow -a_1 \otimes \cdots \otimes a_{k+1} \otimes a_k \otimes \cdots \otimes a_n.$$]

- From tensor multiplication, we clearly inherit the possibility to multiply p-forms (more common notation than n-forms, but clashes with our $p \in M$):

$$\omega_p \wedge \omega_q = \omega_{p+q},$$ defined as the equivalence class of $\omega_p \otimes \omega_q$.

- A natural basis of $(T^{*\otimes p})_A$ is inherited from $T^{*\otimes p}$:

suppress p to avoid confusion with p of ω_p.

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

(equiv. class of $dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p}$).

- The corresponding linear functional is the totally antisymm. element of $T^{*\otimes n}$ in the same equiv. class, e.g.

$$dx^1 \wedge \cdots \wedge dx^p \text{ "="} \frac{1}{p!} \sum_\sigma \operatorname{sgn}(\sigma)\, dx^{\sigma(1)} \otimes \cdots \otimes dx^{\sigma(p)}$$ (σ runs over all permutations)

- This fixes the action on vectors:

$$dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}(v_1,\dots,v_p) = \frac{1}{p!}\sum_{\sigma} \mathrm{sgn}(\sigma)\, v_{\sigma(1)}^{\mu_1}\cdots v_{\sigma(p)}^{\mu_p}$$

↑ all permutations.

- Using our summation convention, we can decompose any p-form in this basis: $\omega_p = \frac{1}{p!}\underbrace{(\omega_p)_{\mu_1\dots\mu_p}}_{\text{totally antisymmetric (tensor field)}} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}$.

- We also have

$$\omega_p\wedge\omega_q = \frac{1}{p!}\frac{1}{q!}(\omega_p)_{\mu_1\dots\mu_p}(\omega_q)_{\nu_1\dots\nu_q}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}\wedge dx^{\nu_1}\wedge\dots\wedge dx^{\nu_q}$$

$$= \frac{1}{(p+q)!}(\omega_p\wedge\omega_q)_{\mu_1\dots\mu_{p+q}}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{p+q}}$$

or $(\omega_p\wedge\omega_q)_{\mu_1\dots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}(\omega_p)_{[\mu_1\dots\mu_p}(\omega_q)_{\mu_{p+1}\dots\mu_{p+q}]}$.

- We are now ready to extend our "exterior derivative" d to higher forms. We require linearity, Leibnitz rule and $d^2 = d\circ d = 0$.

$d\omega_p = \omega_{p+1}$; $d(\omega_p\wedge\omega_q) = d\omega_p\wedge\omega_q + (-1)^p\omega_p\wedge d\omega_q$.

- With the above, we can simply calculate the action of d on any ω_p in a basis:

$$d\omega_p = d\left(\frac{1}{p!}(\omega_p)_{\mu_1\dots\mu_p}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}\right)$$ ← use the above properties!

$$= \frac{1}{p!}\partial_\mu(\omega_p)_{\mu_1\dots\mu_p}\, dx^\mu\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_p}$$

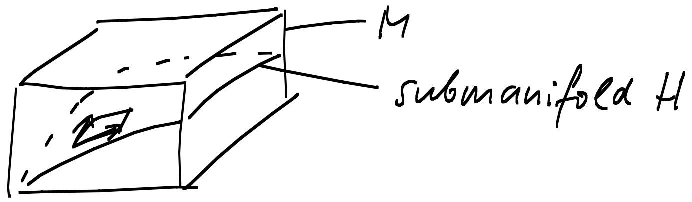
$$= \frac{1}{(p+1)!}\partial_{\mu_1}(\omega_p)_{\mu_2\dots\mu_{p+1}}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_{p+1}}$$

$$\Rightarrow \quad (d\omega_p)_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} (\omega_p)_{\mu_2 \cdots \mu_{p+1}]} \,.$$

[It is easy to check that $d = dx^\mu \partial_\mu$, where dx^μ has to multiplied with "$\wedge$" from the left.]

6.2 Integration with p-forms

- The basic reason behind the above formalism is that a p-form is naturally integrated over p-dim. submanifold:



- Cut it into little parallelepipeds (we will attempt to sum over them shortly; in the spirit of the Riemann integral). Each parallelepiped can be characterized by p linearly independent tangent vectors v_a $(a = 1 \cdots p)$. (Each edge goes from some $x^\mu_{(0)}$ to $x^\mu_{(0)} + v_a{}^\mu$.) To integrate, we need to ascribe a real number to each infinitesimal piece of the surface. While the ordering of the edges does not matter in detail, we can define an <u>orientation</u> by saying which orderings are even and which are odd. Thus, what we are looking for is a linear functional on $v_1 \cdots v_a$ which is only sensitive to the sign of the permutation of the arguments, i.e. a p-form.

- Define (very roughly):

$$\int_H \omega_p = \lim_{\text{each parall.} \to 0} \sum_{\text{all parall.}} \omega_p(v_1, \dots, v_p) \,.$$

- Given ω_p on M, we can define the <u>pullback</u> ω_p^* on the submanifold H.

- Then the integral reads:

$$\int_M \omega_p = \int (\omega_p^*)_{12\cdots p}\, dx^1 dx^2 \dots dx^p \,.$$

(Check reparametrization invariance!)

- Without proof: Stoke's theorem: $\int_M d\omega_p = \int_{\partial M} \omega_{p-1}$

M ↑ p-dim. manifold; ∂M ↑ its boundary (a (p-1)-dim. manifold)

- Given a metric $g_{\mu\nu}$ on p-dim. manifold M, one can show that $V_{\mu_1\cdots\mu_p} = \sqrt{-g}\,\varepsilon_{\mu_1\cdots\mu_p}$ represent the components

(ε ↑ Levi-Civita symbol)

of a tensor. Hence, we have defined a p-form V. This <u>volume form</u> is the natural object for integrating over a metric manifold.

(e.g. $S_{\text{gravit.}} = \frac{M^2}{2}\int V \cdot R$, V ↑ volume form, R ↖ Ricci scalar).

- Using the metric, there is a natural way to assign to every p-form on an n-dim. manifold an (n-p)-form:

$$(*\omega)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\sqrt{-g}\,\varepsilon_{\mu_1\cdots\mu_n}\, g^{\mu_{n-p+1}\nu_1}\cdots g^{\mu_n\nu_p}\,\omega_{\nu_1\cdots\nu_p}$$

(This is referred to as the <u>Poincaré dual</u> of ω or its <u>Hodge star</u>.)

<u>Problem:</u> check that $** = (-1)^{p(n-p)+1}$.

6.3 Electrodynamics with diff. forms

- charged scalar field : φ - complex 0-form
- gauge trf.: $\varphi \to \varphi' = e^{-i\lambda}\varphi$ (with real 0-form λ)
- covariant exterior derivative : $D\varphi = d\varphi + iA\varphi$,
- gauge potential: A – 1-form
- gauge trf. of $D\varphi$: $D\varphi \to D'\varphi' = d(e^{-i\lambda}\varphi) + iA'e^{-i\lambda}\varphi$

$$= e^{-i\lambda}(d - i(d\lambda) + iA')\varphi \underset{\text{if:}}{=} e^{-i\lambda}D\varphi$$

!

- gauge trf. : $A \longrightarrow A' = A + d\lambda$
- field strength: $F = dA$ (F – 2-form)
- action: $S = \frac{1}{2}\int\limits_{\text{space-time}} F\wedge(*F) + \int\limits_{\substack{\text{world-line}\\ \text{of charged particle}}} A$

(Note that the normalizations of the components of A & F are different from the usual formulation. But this is purely conventional.)

- current: j – $(n-1)$-form

(This is clear since one should be able to integrate a current over a $\underbrace{\text{(hyperplane of space)}}_{n-2\text{ dims.}}\times\text{(time interval)}$ to get the amount of charge passing through this plane during a certain interval of time.

Alternatively, in the static case, the integral of j over all space ($n-1$ dims.) should give the total charge.)

- Hence, the "smeared out" version of $\int_{\text{world line}} A$ reads $\int_{\text{space-time}} j \wedge A$.

- Let us now vary the action

$$S = \int \left(\tfrac{1}{2} F \wedge * F + j \wedge A \right) \quad ; \quad F = dA$$

with respect to A:

$$\delta S = \int \left(\tfrac{1}{2} (d\,\delta A) \wedge * F \right) + \tfrac{1}{2} F \wedge * (d\,\delta A) + j \wedge \delta A \Big)$$

Use $\quad F \wedge * \delta F = * F \wedge \delta F$

and $\int d(\cdots) = 0$ (ignoring boundary terms).

$$\delta S = \int \delta A \wedge (d * F - j) \quad \Rightarrow \quad \left. \begin{aligned} d * F &= j \\ dF &= 0 \end{aligned} \right\} \text{ Maxwell eqs.}$$

(The second eq. is an obvious consequence of $d^2 = 0$.)

<u>Comment:</u> More generally, if there were also magnetic charges, we would have

$$d * F = j_{el.}$$

$$dF = j_{mag.}\,.$$

Now it is impossible to find A with $F = dA$. However, if $j_{el} = 0$ and $j_{mag.} \neq 0$, we could declare $*F \equiv \tilde{F}$ to be our fundamental field strength and write

$$d\tilde{F} = 0 \qquad \text{with } \tilde{F} = d\tilde{A}.$$
$$d*\tilde{F} = j_{mag.}$$

(This is the "dual formulation" of electrodynamics)

Note: This is not only a nice way of describing electrodynamics but also a good preparation for string theory, where "higher form fields" ($F_p = dA_{p-1}$; $p > 2$) are ubiquitous.

6.4 Vielbein (tetrad or vierbein in $n=4$)

- In an alternative approach to GR, one starts with a basis of basis vectors at every point of M:

$$\underline{\text{the vielbein } e_a{}^\mu.} \qquad a = 1 \cdots n$$
$$(a = 1..4 \text{ in } 4 \text{ dims.})$$

We can think of them as defining a Lorentz frame at every point. (Note however, that they are in general not the basis vectors of any coordinate system, not even in a small neighbourhood. If they were, spacetime would be flat, as we will see shortly.)

- The index a is called a local Lorentz-index (as opposed to the Einstein- or coordinate index μ).

- We can also introduce the inverse vielbein $e_a{}^\mu$ defined by

$$e_a{}^\mu e^b{}_\mu = \delta_a{}^b \quad \text{or, equivalently,} \quad e_a{}^\mu e^a{}_\nu = \delta^\mu{}_\nu.$$

(If we wish, we can also lower or raise the Lorentz index with η_{ab}, writing e.g. $e^{a\mu} = \eta^{ab} e_b{}^\mu$ or $e_{a\mu} = \eta_{ab} e^b{}_\mu$.)

- We can think of the inverse vielbein as a basis of 1-forms.
- Most importantly, given a vielbein, we can define a metric as
$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu .$$
Multiplying with the inverse vielbein,
$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab} ,$$
we see that our T_p-basis is orthonormal in this metric.
- Multiplying both sides with one inverse vielbein,
$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu e^b{}_\rho = \eta_{ab} e^b{}_\rho$$
$$g_{\mu\rho} e_a{}^\mu = \eta_{ab} e^b{}_\rho ,$$
we see that vielbein and inverse vielbein are connected by raising (or lowering) indices with η_{ab} and $g_{\mu\nu}$.
- Since we still want only the metric to be a physical d.o.f., we introduce a gauge transformation (local Loretz rotations), eliminating the extra information we have introduced at the beginning:
$$e^a{}_\mu \to e'^a{}_\mu = \underset{\substack{\uparrow \\ \text{fct. on } M.}}{\Lambda^a{}_b} e^b{}_\mu$$

(We declare two configurations linked by such a trf. to be physically equivalent.)

- Clearly, given a vielbein, any tensor can be written with Lorentz indices only:

$$t^{\mu_1 .. \mu_m}{}_{\nu_1 .. \nu_n} \leftrightarrow t^{a_1 .. a_m}{}_{b_1 .. b_n}$$

$$= e^{a_1}{}_{\mu_1} .. e^{a_m}{}_{\mu_m} e_{b_1}{}^{\nu_1} ... e_{b_n}{}^{\nu_n} t^{\mu_1 .. \mu_m}{}_{\nu_1 .. \nu_n} .$$

- It would appear that we can now easily take derivatives of vector fields since, e.g.

$v^a = e^a{}_\mu v^\mu$ is just a fct. and we could try to consider its derivative $\partial_\mu v^a$ (which is indeed a tensor in our old way of thinking).

- However, this does not transform covariantly under local Lorentz rotations,

$$\partial_\mu v_a' = \partial_\mu (\Lambda_a{}^b v_b) \neq \Lambda_a{}^b \partial_\mu v_b' ,$$

as would be desirable.

- It is useful to reformulate this problem in our diff.-form language:

$$d v_a = (\partial_\mu v_a) dx^\mu$$

$$v_a \rightarrow \Lambda_a{}^b v_b \quad ; \quad dv_a \not\rightarrow \Lambda_a{}^b dv_b .$$

6.5 Covariant exterior derivative & curvature

- We thus need to make d covariant, which can be done using a connection 1-form:

$$\underset{\uparrow}{D v_a \equiv dv_a} + \underset{\uparrow}{\omega_a{}^b v_b} \qquad (\omega_a{}^b = (\omega_\mu)_a{}^b dx^\mu)$$

both terms are 1-forms.

- As in electrodynamics, we now prescribe gauge trf. properties to $\omega_a{}^b$ ensuring that Dv_a transforms like v_a under local Lorentz rotations. We supress indices and think of Λ & ω as matrices:

$$Dv \to D'v' = (d+\omega')(\Lambda v) = d\Lambda v + \Lambda dv + \omega'\Lambda v$$

$$\overset{!}{=} \Lambda Dv = \Lambda dv + \Lambda \omega v$$

$$\Rightarrow \quad d\Lambda + \omega'\Lambda = \Lambda\omega \quad \text{or} \quad \omega' = \Lambda\omega\Lambda^{-1} - d\Lambda\Lambda^{-1}$$

- <u>Note:</u> The last ("inhomogeneous") term in this trf.-law contributes $-\partial_\mu\Lambda\Lambda^{-1}dx^\mu$ to ω'.

  We can think of a varying Λ e.g. as $e^{\Sigma t}\Lambda$ near $t=0$ (with $\Lambda \in \mathrm{lie}(SO(1,3))$, i.e., "generators of $SO(1,3)$").

  Thus, $\partial_t(e^{\Sigma t}\Lambda)\Lambda^{-1}\big|_{t=0} = \Sigma$.

  This motivates the definition $\omega_\mu \in \mathrm{lie}(SO(1,3))$

  or : <u>ω is an $\mathrm{lie}(SO(1,3))$-valued 1-form.</u>

- To check equivalence to our previous discussion of connections and covariant derivatives, we can always write $D = dx^\mu D_\mu$, thereby defining what D_μ means.

Given any tensor, we can now first write it with Lorentz indices, apply this D_μ, and then go back to Einstein indices. (For explicit consistency checks of this type, see problems.)

- We have not yet demanded the consistency of the connection the metric (or vielbein).

- We define: $-T^a = De^a$ where $e^a = e^a{}_\mu dx^\mu$

  Thus, the torsion is, in this approach, a Lorentz-vector-valued 2-form.

- For GR, we demand, as before, vanishing torsion, i.e., covariant constancy of the vielbein:

  $$0 = De^a = de^a + \omega^a{}_b \wedge e^b$$

  (as before, this fixes ω)

- The curvature is defined as

  $$R = D \circ D = (d+\omega)(d+\omega)$$

  $$= d \circ d + d \circ \omega + \omega d + \omega \wedge \omega$$

  $$= 0 + (d\omega) - \omega d + \omega d + \omega \wedge \omega$$

  $$\boxed{R = d\omega + \omega \wedge \omega} \quad \left(\begin{array}{c} \text{Lie}(SO(1,3)) - \text{valued} \\ 2-\text{form} \end{array} \right)$$

  Wedge product of forms together with matrix-multiplication of two Lie(SO(1,3))-matrices

[It is a nice exercise to check this is consistent with

$$[D_\mu, D_\nu] v_a = (R_{\mu\nu})_a{}^b v_b \quad (\text{with } D_\mu v_a = \partial_\mu v_a + (\omega_\mu)_a{}^b v_b)$$

and this is, in turn, consistent with our earlier definition]

Final comments:

- At a technical level, we will see in the derivation of the Schwarzschild-solution that it is sometimes easy to "guess" $\omega_a{}^b$ from $de_a + \omega_a{}^b e_b = 0$ and then calculate R from $R_a{}^b = d\omega_a{}^b + \omega_a{}^c \wedge \omega_c{}^d$.

- At a conceptual level, the vielbein / differential-form approach has the advantage of showing clearly that GR is really just a gauge theory with gauge group $SO(1,3)$. [This will be even more impressive when you learn about non-abelian gauge theories in the QFT course.]

- Finally, Bianchi-identities are very easy in this context:

1st: $DR = 0$ (corresponds to $D_{[\mu} R_{\nu\rho]\sigma}{}^{\tau} = 0$ above)

(derivation: $DR = dR + \omega \wedge R - R \wedge \omega$ ← need to act on both indices of R!

$$= d(d\omega + \omega\wedge\omega) + \omega\wedge R - R\wedge\omega$$

$$= d\omega\wedge\omega - \omega\wedge d\omega + \omega\wedge d\omega + \omega\wedge\omega\wedge\omega - d\omega\wedge\omega - \omega\wedge\omega\wedge\omega$$

$$= 0.)$$

2nd: $(Re)^a = 0$ (corresponds to $R_{\mu\nu\rho}{}^{\sigma} + (\text{cycl. perms in } \mu\nu\rho) = 0$)

(This is obvious since $De^a = 0$.)

recall: $e_b = e_{b\mu} dx^\mu$ etc.

To derive the component form: $(Re)_a = R_a{}^b e_b$;
$(Re)^a = -R_b{}^a e^b$ (follows since $e_a e^a$ is a scalar)
$\Rightarrow (Re)^a = -R_b{}^a e^b = -e^c e^d R_{cdb}{}^a e^b = 0 \Rightarrow$ antisymm. of cdb gives zero.