7. The Schwarzschild Solution

7.1 Static, spherically symmetric solutions

- A spacetime is called <u>stationary</u>, if there exists a one-parameter group of isometries ϕ_t $(t \in \mathbb{R})$ with only time-like orbits.

<u>Comment:</u> Recall that an isometry is a diffeomorphism $\phi : M \to M$ respecting the metric. More generally, any one-parameter group of diffeomorphisms ϕ_t (not necessarily isometries) defines a vector field:

For any $p \in M$ $v(p)$ is the tangent vector of the curve $\phi_t(p)$ $\quad (\phi_{..}(p) : \mathbb{R} \to M,\ t \mapsto \phi_t(p))$ at the point p. These are called integral curves of ϕ_t.

(<u>Note:</u> one-parameter <u>group</u> simply means that
$$\phi_t \circ \phi_s = \phi_{t+s} \quad \& \quad \phi_0 = id.)$$

Conversely, any smooth vector field has a unique family of integral curves (curves tangent to the vector field at every point). These curves define a diffeomorphism ϕ_t where $\phi_t(p)$ is the point lying a parameter-distance t away from p on the integral curve through p.

If ϕ_t is an isometry, the associated vector field is called a <u>Killing vector field</u> (usually ξ^μ).

<u>Fact:</u> ξ Killing field $\iff$ $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$

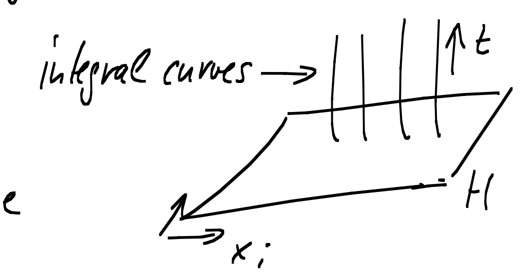
(Killing equation)

(without proof)

- We require the stronger condition that our spacetime is <u>static</u>, which means that $\exists$ a spacelike hypersurface H orthogonal to ξ.

- If so, we can parametrize H by x^i $(i = 1 \cdots 3)$ and carry it forward (or "backward") using the isometry ϕ_t. Then M will be parameterized by $(t, x^1, \cdots, x^3)$.

- Since the integral curves are orthogonal to H (and, because we use an isometry, also to all its images $H_t \equiv \phi_t(H)$) ds^2 will not have any terms $\sim dt\, dx^i$:



$$ds^2 = -f(\bar{x})\, dt^2 + h_{ij}(\bar{x})\, dx^i dx^j$$

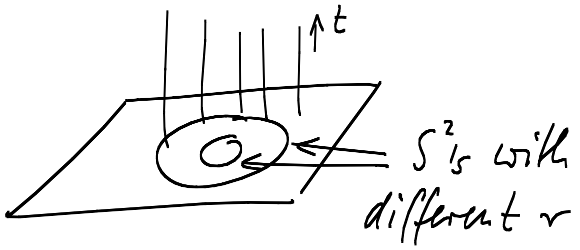
independent of t since t parametrizes a family of isometries

(<u>Note</u>: A stationary but not static spacetime would have contributions $\sim dt\, dx^i$ breaking the symmetry $t \to -t$. Physically, one may think e.g. of a streaming fluid, which is stationary but not static.)

- A spacetime is <u>spherically symmetric</u> if it has an isometry group $SO(3)$ the orbits of which are S^2's. We can parametrize each such orbit by θ, φ with metric

$$ds^2 = r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad \text{(by } SO(3) \text{ isometry).}$$

- We also assume that the time-like Killing vector field ξ is unique. Then its projection on any of the S^2's vanishes (or SO(3) symm. would be broken). Thus each S^2 lies completely within one of the H_t's. Unless all S^2 have the same r, we can use r as the 3rd coordinate to parametrize the H_t's:



- We carry the spherical coordinates θ, φ from one S^2 to the neighbouring S^2's by geodesics in H orthogonal to S^2.

This ensures, when we vary r but not θ, φ, we move orthogonally to S^2, i.e. no terms $dr\,d\varphi$ or $dr\,d\theta$ in the metric.

$\Rightarrow$ most general metric:

$$ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

7.2 Derivation of the Riemann tensor using the vielbein

From $g_{\mu\nu} = e_{a\mu}\eta^{ab}e_{b\nu}$ we see that (up to the factor η), the vielbein is just the square root of the metric $g_{\mu\nu}$. Thus, we can set

$$e_{a\mu} = \mathrm{diag}\left(f^{1/2}, h^{1/2}, r, r\sin\theta\right)$$

in a coordinate system $\{x^\mu\} = \{t, r, \theta, \varphi\}$.

More explicitly, we have
$$\begin{aligned} e_0 &= f^{1/2}\,dt \quad \left(\Rightarrow (e_1)_\mu = f^{1/2}(dt)_\mu = f^{1/2}\delta_{1\mu}\right) \\ e_1 &= h^{1/2}\,dr \\ e_2 &= r\,d\theta \\ e_3 &= r\sin\theta\,d\varphi . \end{aligned}$$

We now want to "guess" the connection 1-forms $\omega_a{}^b$ from

$$de_a + \omega_{ab}\, e^b = 0 .$$

(Note that here the form indices are suppressed. ω_{ab} is the lower-index form of $\omega_a{}^b \in \mathrm{Lie}(SO(1,3))$, which implies that ω_{ab} is antisymmetric in a, b.
<u>Problem:</u> proof the antisymmetry of ω_{ab}.)

$a = 0$ $\quad de_0 + \omega_{01} e^1 + \omega_{02} e^2 + \omega_{03} e^3 = 0$

$$de_0 = d\left(f^{1/2}(r)\,dt\right) = \tfrac{1}{2} f' f^{-1/2}\, dr\, dt \quad (\text{suppressing "}\wedge\text{"})$$

$$\Rightarrow \tfrac{1}{2} f' f^{-1/2}\, dr\, dt + \omega_{01} h^{1/2} dr + \omega_{02}\, r\, d\theta + \omega_{03}\, r \sin\theta\, d\varphi = 0$$

guess: $\omega_{02} = \omega_{03} = 0$; $\omega_{01} = \frac{1}{2} f'(fh)^{-1/2} dt + \underset{\substack{\uparrow \\ \text{undetermined since } dr\,dr = 0}}{\alpha_1}\, dr$

$a = 1$ $\quad de_1 + \omega_{10} e^0 + \omega_{12} e^2 + \omega_{13} e^3 = 0$

$$de_1 = d\left(h^{1/2}(r)\, dr\right) = 0$$

$$\Rightarrow -\omega_{10} f^{1/2} dt + \omega_{12}\, r\, d\theta + \omega_{13}\, r \sin\theta\, d\varphi = 0 \quad \Rightarrow \quad \alpha_1 = 0$$

ansatz: $\omega_{12} = \alpha_2\, d\theta + \alpha_3\, d\varphi$; $\omega_{13} = \alpha_4\, d\varphi + \alpha_3 (\sin\theta)^{-1} d\theta$

$a = 2$ $\quad de_2 + \omega_{20} e^0 + \omega_{21} e^1 + \omega_{23} e^3 = 0$

$$de_2 = d(r\, d\theta) = dr\, d\theta$$

$$\Rightarrow dr\, d\theta - \omega_{20} f^{1/2} dt + \omega_{21} h^{1/2} dr + \omega_{23}\, r \sin\theta\, d\varphi = 0$$

Using our ansatz for ω_{12} and our guess $\omega_{20} = 0$ we find:

$$\alpha_2 = -h^{-1/2} \quad ; \quad \omega_{23} = -\alpha_3 \frac{h^{1/2}}{r \sin\theta}\, dr + \alpha_5\, d\varphi$$

$(a=3)$ $\quad de_3 + \omega_{30} e^0 + \omega_{31} e^1 + \omega_{32} e^2 = 0$

$$de_3 = d(r\sin\theta\, d\varphi) = r\cos\theta\, d\theta d\varphi + \sin\theta\, dr d\varphi$$

$$\Rightarrow r\cos\theta\, d\theta d\varphi + \sin\theta\, dr d\varphi - \omega_{30} f^{1/2} dt + \omega_{31} h^{1/2} dr + \omega_{32}\, r d\theta = 0$$

Use our guess $\omega_{30} = 0$ and our ansatz for ω_{13}, ω_{23}.

$$\Rightarrow \quad \alpha_4 = -h^{-1/2}\sin\theta \quad ; \quad \alpha_3 = 0 \quad ; \quad \alpha_5 = -\cos\theta$$

- Since we now have solved all 4 equations, and since ω is unambiguously determined by $de + \omega e = 0$, we now know that the non-zero components of ω are:

$$\left\| \begin{aligned} \omega_{01} &= \tfrac{1}{2} f' (fh)^{-1/2}\, dt \\ \omega_{12} &= -h^{-1/2}\, d\theta \\ \omega_{13} &= -h^{-1/2} \sin\theta\, d\varphi \\ \omega_{23} &= -\cos\theta\, d\varphi \end{aligned} \right\|$$

- We now compute the Riemann tensor from $R = d\omega + \omega\wedge\omega$

or $\quad R_{ab} = d\omega_{ab} + \omega_{ac}\wedge\omega_{db}\, \eta^{cd}$: (form indices suppressed)

$$R_{01} = \frac{d}{dr}\left[\tfrac{1}{2} f'(fh)^{-1/2}\right] dr dt$$

$$R_{02} = -\tfrac{1}{2} f' f^{-1/2} h^{-1}\, dt d\theta$$

$$R_{03} = -\tfrac{1}{2} f' f^{-1/2} h^{-1} \sin\theta\, dt d\varphi$$

$$R_{12} = \tfrac{1}{2} h' h^{-3/2}\, dr\, d\theta$$

$$R_{13} = \tfrac{1}{2} h' h^{-3/2} \sin\theta\, dr d\varphi$$

$$R_{23} = (1 - h^{-1}) \sin\theta\, d\theta d\varphi$$

- From this, we can immediately read off the components in the form $R_{\mu\nu ab}$, e.g.

$$R_{\mu\nu 12} = \frac{1}{2} h' h^{-3/2} (dr\, d\theta)_{\mu\nu} \Rightarrow \text{the only non-zero components are}$$

$$R_{1212} = - R_{2112} = \frac{1}{2} h' h^{-3/2}$$

- The Ricci tensor follows from contracting the 2nd & 4th index, i.e.

$$R_{\mu a} = R_{\mu\nu ab}\, e^{b\nu} \quad \text{or} \quad R_{ca} = R_{\mu\nu ab}\, e^{b\nu} e_c{}^{\mu}$$

This gives (with both indices being Lorentz indices)

$$R_{00} = \left(R_{0101} e^{11} + R_{0202} e^{22} + R_{0303} e^{33}\right) e_0{}^0$$

$$= \Big[-\frac{d}{dr}\left(\frac{1}{2} f' (fh)^{-1/2}\right) h^{-1/2} - \frac{1}{2} f' f^{-1/2} h^{-1} r^{-1} - \frac{1}{2} f' f^{-1/2} h^{-1} \sin\theta \cdot$$

$$\cdot (r \sin\theta)^{-1} \Big] \left(- f^{-1/2}\right)$$

$$= \frac{1}{2} (fh)^{-1/2} \frac{d}{dr}\left(f' (fh)^{-1/2}\right) + (fhr)^{-1} f' \qquad (1)$$

analogously one finds (→ problems)

$$R_{11} = -\frac{1}{2} (fh)^{-1/2} \frac{d}{dr}\left(f' (fh)^{-1/2}\right) + r^{-1} h^{-2} h' \qquad (2)$$

$$R_{22} = R_{33} = -\frac{1}{2} (rfh)^{-1} f' + \frac{1}{2} r^{-1} h^{-2} h' + r^{-2} (1 - h^{-1}) \qquad (3)$$

This equality and the vanishing of all off-diagonal components can also be derived from symmetry considerations.

7.3 Schwarzschild solution

- We are now ready to discuss the Einstein eq.

$$\bar{M}_p^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = T_{\mu\nu} \qquad (\bar{M}_p \text{ reduced Planck mass})$$

- Taking the trace, we find $-M_p^2 R = T \quad (T \equiv T_\mu{}^\mu)$, which allows us to rewrite the Einstein eq. as

$$M_p^2 R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T .$$

- Since we are interested in spherically symmetric vacuum solutions (i.e. the gravit. field <u>outside</u> a spherically symmetrical body) we set $T_{\mu\nu} = 0$ ($\Rightarrow T = 0$) and find

$$R_{00} = 0 \qquad (1')$$
$$R_{11} = 0 \qquad (2')$$
$$R_{22} = 0 \qquad (3')$$

- $(1') + (2')$ reads $\quad (fhr)^{-1} f' + r^{-1} h^{-2} h' = 0$

or $\quad f'/f + h'/h = 0$

or $\quad (\ln f + \ln h)' = 0$

or $\quad f \cdot h = \text{const.}$

- By rescaling t we can set const. $= 1$ or $h = f^{-1}$.
- Using $h = f^{-1}$ & $h'/h = -f'/f$, $(3')$ takes the form

$$-\frac{1}{2} r^{-1} f' - \frac{1}{2} r^{-1} f' + r^{-2}(1-f) = 0$$

$$-rf' + 1 - f = 0$$

$$rf' + f = 1$$

$$(rf)' = 1$$

$$\Rightarrow fr = r + c$$

$$f = 1 + \frac{c}{r}$$

- Obviously, the single integration constant we found has to encode the information about the mass of the central body. The precise relation (to be derived shortly) reads

$$C = 2MG_N = \frac{2M}{M_p^2} = \frac{2M}{8\pi \bar{M}_p^2} \qquad (\hbar = c = 1)$$

- It is convenient to use units where $G_N = M_p^{-1} = 1$ (do not confuse with the frequently used units where $\bar{M}_p = 1$).

$\Rightarrow$ Schwarzschild metric:

$$\left\| \; ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \underbrace{d\Omega^2}_{S^2\text{-metric}} \; \right\|$$

<u>Comments:</u>

- At $r \to \infty$ we obviously recover flat Minkowski space.
- At large r we find we find the Newtonian potential of a spherical body (but in a different gauge than in our previous discussion of the Newtonian limit).

- There are metric singularities at $r = 2M$ and at $r = 0$. The first is a mere coordinate singularity (can be removed by reparametrization), the second is a curvature singularity (it is physical).

- If we had dropped the staticity requirement, we had still found (with much more work) only this solution. (Birkhoff's theorem: All spherically symmetric spacetimes without matter are static.)

- It is possible to repeat the analysis with $T_{\mu\nu} \neq 0$

describing an ideal fluid (relevant, e.g., for the interiour of a star)

7.4 Gravitational redshift

- Light rays travel on light-like geodesics. Recall the metric
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 .$$

- A light ray emitted at r_1 without angular component of 4-momentum (only dt & dr with $dr/dt = (1-2M/r)$) will not develop an angular component subsequently because of spherical symmetry ($d\varphi$ & $d\theta$ will stay zero).

- Wave maxima emitted at t_1 & $t_1+\Delta t$ will arrive at r_2 at times t_2 and $t_2+\Delta t$ (since the second geodesic is simply the first shifted by Δt, according to the time-like isometry $t \to t+\Delta t$).

- Thus, the ratio of frequencies (ω_1 at t_1 & ω_2 at t_2) is
$$\frac{\omega_2}{\omega_1} = \frac{\sqrt{g_{00}(r_1)\,\Delta t^2}}{\sqrt{g_{00}(r_2)\,\Delta t^2}} = \sqrt{\frac{1-2M/r_1}{1-2M/r_2}} .$$

$\Rightarrow$ The frequency of a light-ray climbing out of the gravitational potential ($r_2 > r_1$) gets reduced.

- In particular, if the emitter approaches $r_1 = 2M$, the frequency observed at any $r > 2M$ approaches zero.

(No light can be emitted from $r \leq 2M$. This is where the black hole horizon is; to be discussed in more detail later on.)

7.5 Perihelion shift

- We will now consider the motion of point-masses in the Schwarzschild geometry. (In fact, most of the analysis is easily done for general $f(r), h(r)$ wich leads to constraints on alternative theories of gravitation.)

- Consider some Killing vector field ξ^μ and a geodesic $u^\mu(\tau)$. We have

$$u^\mu D_\mu (u^\nu \xi_\nu) = u^\mu u^\nu D_\mu \xi_\nu \underset{\uparrow}{=} - u^\mu u^\nu D_\nu \xi_\mu = - u^\nu u^\mu D_\mu \xi_\nu$$

by the Killing eq. $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$

$$\Rightarrow \quad u^\mu D_\mu (u^\nu \xi_\nu) = u^\mu \partial_\mu (u^\nu \xi_\nu) = 0$$

$\Rightarrow$ $u^\nu \xi_\nu$ is constant along the geodesic u^ν.

- Now let $u^\mu(\tau)$ be time-like (τ is the eigentime) and $\xi^\mu = (1,0,0,0)$ be the time like Killing vector field of the Schwarzschild solution.

- Define $E = -m\, g_{\mu\nu} \xi^\mu u^\nu = m f(r) u^0 = m f(r) \frac{dt}{d\tau} = m\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$

  We have $E = \text{const.}$ and we interpret it as the energy of the particle. [This is justified since, if the trajectory extends to $r = \infty$, we find $E = m u^0$, i.e., the well-known special-relativistic kinetic energy.]

- Analogously, let $\varphi^\mu = (0,0,0,1)$ be the Killing vector field corresponding of rotations around the z-axis.

  (Recall that the metric is independent of $x^3 = \varphi$.)

- Define the second conserved quantity

$$L_z = m\, g_{\mu\nu} \psi^\mu u^\nu = m\, r^2 \sin^2\theta \frac{d\varphi}{d\tau},$$

the "angular momentum in z-direction".

- Using rotational symmetry, any initial position and velocity vector can be brought to respect $\theta(\tau_0) = \pi/2$ & $\dot\theta(\tau_0) = 0$ [Position & velocity in the equatorial plane]. Due to the reflection symmetry

$$\theta \to \pi - \theta \quad \text{or} \quad \theta - \pi/2 \to -(\theta - \pi/2)$$

of the metric, the complete trajectory will then lie in this plane. $\Rightarrow$ We can set $\sin\theta = 1$.

- To simplify formulae, we redefine $E, L \to E/m, L/m$ and use $d/d\tau(\cdot\cdot) = (\cdot\cdot)^\cdot$.

$$\Rightarrow \boxed{E = f(r)\,\dot t} \; ; \; \boxed{L = r^2 \dot\varphi} \text{ are conserved.}$$

- In addition, we have $g_{\mu\nu} u^\mu u^\nu = -1$, which implies

$$\boxed{-f(r)\,\dot t^2 + h(r)\,\dot r^2 + r^2 \dot\varphi^2 = -1}$$

$\downarrow$ use defs. of E & L

$$-E^2 f(r)^{-1} + h(r)\,\dot r^2 + \frac{L^2}{r^2} = -1$$

$$\frac{1}{2}\dot r^2 + \frac{1}{2h(r)}\left(\frac{L^2}{r^2} + 1 - \frac{E^2}{f(r)}\right) = 0$$

- This is equivalent to a 1-dim. mechanics problem for a particle with energy zero and potential

$$V(r) = \frac{1}{2h(r)}\left(\frac{L^2}{r^2} + 1 - \frac{E^2}{f(r)}\right) = -\frac{E^2}{2} + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + 1\right)$$

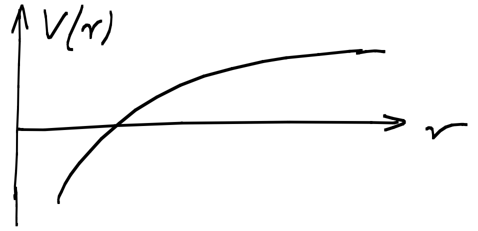
$$\Rightarrow V(r) = \frac{1}{2}(1-E^2) - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

this term confirms that $C = 2M$ was the correct identification.

$$V'(r) = \frac{M}{r^4}\left(r^2 - (L^2/M)r + 3L^2\right)$$

• The extrema of $V(r)$ are at $r_{1,2} = \frac{1}{2M}\left(L^2 \pm \sqrt{L^4 - 12L^2M^2}\right)$

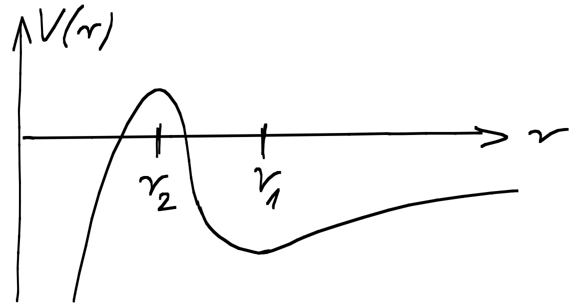
$\Rightarrow$ ① $L^2 < 12M^2$ – no (real) extrema:



if $\dot{r} \le 0$, the particle always "falls in".

(if $E^2 < 1$, the particle falls in even we start with $\dot{r} > 0$)

$\Rightarrow$ ② $L^2 > 12M^2$ – max. at r_2 ; min. at r_1

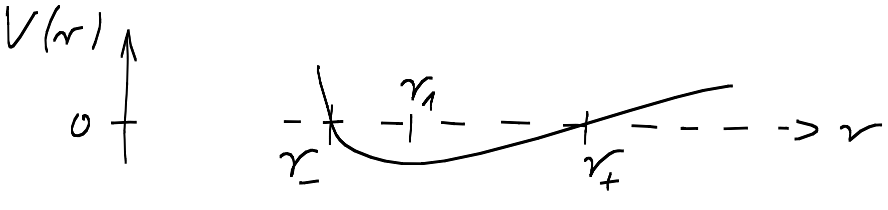


possible motions include: (for appropriate choice of E)

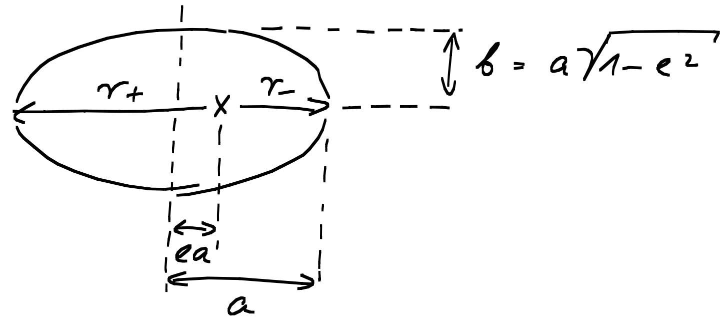
– unstable circular orbit at $r = r_2$

– stable circular orbit at $r = r_1$

– periodic motion near the minimum at r_1
(the GR-generalization of the well-known elliptic trajectories of Newt. gravity)



- Picture of the "would-be ellipse" (or the ellipse arising in the Newtonian limit) for this motion:



- As we will now see, in GR the trajectory does not "close" after one full orbit ($\varphi \to \varphi + 2\pi$). To demonstrate this explicitly, we will need $\varphi(r)$ [in addition to $r(\tau)$, which we have, at least in principle, already obtained by setting up the equivalent 1-dim. problem].

- As in class. mech., $\varphi(r)$ follows from $L = r^2\dot\varphi$ & $\frac{1}{2}\dot r^2 + V(r) = 0$ by eliminating $d\tau$:

$$d\varphi = \frac{L}{r^2}\,d\tau = \frac{L}{r^2}\cdot\frac{dr}{\sqrt{-2V(r)}}$$

- Thus, the particle progresses in φ during a full period in r by

$$2\int_{r_-}^{r_+}\frac{L\,dr}{r^2\sqrt{-2V(r)}}\,.$$

- In the Newtonian case, this would equal 2π. We want to calculate (in leading order in G_N):

$$\Delta\varphi = 2\int_{r_-}^{r_+}\frac{L\,dr}{r^2\sqrt{-2V(r)}} - 2\pi\,.$$

This calculation is possible but painful (see Weinberg's Book).

- I find it more elegant to work with the equivalent diff. eq.:

$$\frac{1}{2}\dot{r}^2 + V(r) = 0 \quad \Rightarrow \quad \frac{1}{2} r'^2 \frac{L^2}{r^4} + V(r) = 0$$

- Let $u = \frac{1}{r}$; $r' = -u'/u^2$

$$\Rightarrow \frac{1}{2}L^2 u'^2 + \frac{1}{2}(1-E^2) - Mu + \frac{1}{2}L^2u^2 - ML^2u^3 = 0$$

$$u'^2 + u^2 = \frac{E^2 - 1}{L^2} + \frac{2M}{L^2}u + 2Mu^3$$

$$\frac{d}{d\varphi}(\ldots) \ \& \ \frac{1}{2u'}(\ldots) \quad \Rightarrow \quad \underbrace{u'' + u = \frac{M}{L^2}}_{} + 3Mu^2$$

This part has the well-known solution

$$u(\varphi) = \frac{1}{p}(1 + e\cos\varphi) \quad \text{with } p = \frac{L^2}{M} = a(1-e^2).$$

- Since we are only interested in the leading correction, we can use this solution in the perturbing term. We need to solve:

$$u'' + u = \frac{M}{L^2} + 3\frac{M^3}{L^4}(1 + e\cos\varphi)^2 .$$

To do that, we separately solve

$$u'' + u = A \quad \Rightarrow \quad u = A$$

$$u'' + u = B\cos\varphi \quad \Rightarrow \quad u = \frac{1}{2}B\varphi\sin\varphi$$

$$u'' + u = C\cos^2\varphi \quad \Rightarrow \quad u = \frac{1}{2}C - \frac{1}{6}C\cos 2\varphi$$

(The first two are obvious; check the last:

$$u'' + u = \frac{2}{3}C\cos 2\varphi + \frac{1}{2}C - \frac{1}{6}C\cos 2\varphi = \frac{1}{2}C(1 + \cos 2\varphi)$$

$$= \frac{1}{2}C(\cos^2\varphi + \sin^2\varphi + \cos^2\varphi - \sin^2\varphi) = C\cos^2\varphi.)$$

- Combining these 3 solutions and the solution $\frac{M}{L^2} e \cos\varphi$ of the homogeneous equation, we find:

$$u = \frac{M}{L^2}(1 + e\cos\varphi) + \frac{3M^3}{L^4}\left(1 + \frac{e^2}{2} - \frac{e^2}{6}\cos 2\varphi + e\varphi\sin\varphi\right)$$

- There is obviously an extremum of u (and hence of r) at $\varphi = 0$ (in fact a perihelion).

- To find the next perihelion, we have to solve $u' = 0$ for φ near 2π:

$$u' = -\frac{Me}{L^2}\sin\varphi + \frac{3M^3}{L^4}\left(\frac{e^2}{3}\sin 2\varphi + e\sin\varphi + e\varphi\cos\varphi\right) \overset{!}{=} 0$$

- Let $\varphi = 2\pi + \Delta\varphi$ ($\Delta\varphi \ll 1$) and use also $\frac{M^3}{L^4} \ll \frac{M}{L^2}$:

$$-\frac{Me}{L^2}\Delta\varphi + \frac{3M^3}{L^4}\cdot e \cdot 2\pi = 0$$

$$\Delta\varphi = \frac{6\pi M^2}{L^2} = \frac{6\pi M}{a(1-e^2)}$$

- Of course M originally appeared together with G_N (which we had set to 1) so that really: $\Delta\varphi = \frac{6\pi G_N M}{a(1-e^2)}$.

- Recalling from mechanics that $T = 2\pi\sqrt{\frac{a^3}{MG_N}}$, we find the "frequency" of the perihelion advance: $\omega_{per.} = \frac{\Delta\varphi}{T} = \frac{3(G_N M)^{3/2}}{a^{5/2}(1-e^2)}$

- <u>Note</u>: We have followed Straumann's book.
  A much simpler derivation for $e = 0$ is given by Wald.

For Mercury: $\omega_{per} \simeq 43''/100$ years (without effect of Venus).
($e \simeq 0.2$)

7.6 Bending of light

- Consider now light-like geodesics in the Schwarzschild geometry. The derivation of the conserved quantities

$$E = f(r)\,\dot t \quad \text{and} \quad L = r^2 \dot\varphi$$

is completely unchanged (but now, of course, τ is not any more the eigentime but the "affine parameter", i.e. a parameter ensuring $u^\mu D_\mu (u^\nu) = 0$).

- But instead of $g_{\mu\nu} u^\mu u^\nu = -1$, we now have

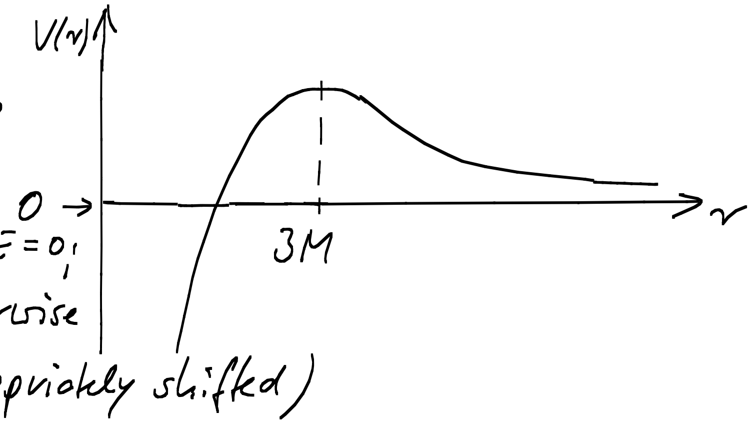
$$g_{\mu\nu} u^\mu u^\nu = -f(r)\,\dot t^2 + h(r)\,\dot r^2 + r^2 \dot\varphi = 0$$

$\Downarrow$ using defs. of E & L

$$\frac{1}{2}\dot r^2 + \frac{1}{2h(r)}\left(\frac{L^2}{r^2} - \frac{E^2}{f(r)}\right) = \boxed{\frac{1}{2}\dot r^2 + V(r) = 0}$$

with $V(r) = -\dfrac{E^2}{2} + \dfrac{L^2}{2r^3}(r-2M)$; $V'(r) = \dfrac{L^2}{2r^4}(-2r+6M)$

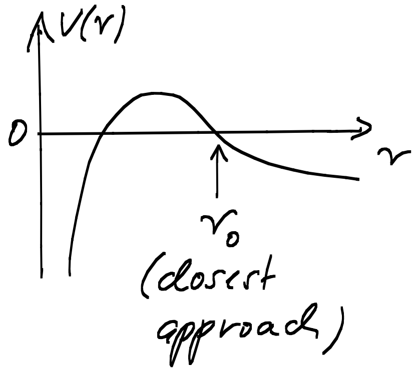
$\Rightarrow$

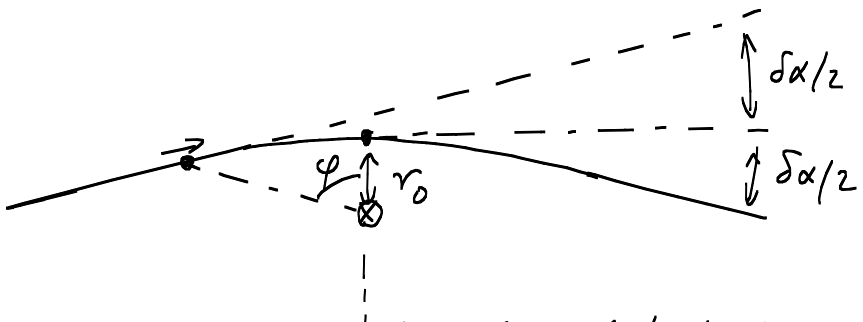


(for $E=0$, otherwise appropriately shifted)

- Obviously, for sufficiently large E the maximum will occur at $V(3M) < 0$, i.e., a light ray coming from infinity will go over the barrier and be absorbed. The critical ratio $(L/E)_c$ corresponds to a critical impact parameter $b_c = (L/E)_c$, which gives an absorption cross section $\sigma = 27\pi M^2$. (Problem: Check these statements!)

- If a light ray is not absorbed, i.e. $V(3M) > 0$, the trajectory looks something like this:





symmetry plane of trajectory $\Rightarrow$ change of angle between $r = r_0$ and $r = \infty$ is $1/2$ of total change

- Thus: $\delta\alpha = \delta\varphi - \pi$;

$$\delta\varphi = 2 \int_{r_0}^{\infty} \frac{L\, dr}{r^2 \sqrt{-2V(r)}}$$

(exactly as for massive case: see Sect. 7.5)

$u = 1/r$

$$\delta\varphi = 2 \int_0^{1/r_0} \frac{L\, du}{\sqrt{E^2 - \frac{L^2}{r^3}(r - 2M)}} = 2 \int_0^{1/r_0} \frac{du}{\sqrt{\underbrace{(E/L)^2}_{= b^{-2}} - u^2 + 2Mu^3}}$$

$= b^{-2}$ from the problem given earlier

- In flat space $(M = 0)$ we have $r_0 = b$ and hence

$$\delta\varphi = 2 \int_0^{1/b} \frac{du}{\sqrt{b^{-2} - u^2}} = 2 \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \left\{ \begin{array}{l} x = \cos\theta \\ dx = -\sin\theta\, d\theta \end{array} \right\}$$

$$= 2 \int_{\pi/2}^{0} (-d\theta) = \pi ,$$ which is clearly the correct answer.

- We now want to include $M \neq 0$ (but working only at leading order in M).

- r_0 is the point where the denominator (i.e. $V(r)$) vanishes:

$$b^{-2} - u_0^2 + 2Mu_0^3 = 0$$

$$\Rightarrow \quad b^{-2} = u_0^2 - 2Mu_0^3 \qquad (u_0 = 1/r_0)$$

- Hence, we need

$$\delta\varphi(M) = 2\int_0^{u_0} \frac{du}{\sqrt{u_0^2 - 2Mu_0^3 - u^2 + 2Mu^3}}$$

$$= \pi + M\left(\frac{d\,\delta\varphi}{dM}\bigg|_{M=0}\right) = \pi + 2M\int_0^{u_0} du \; \frac{-\frac{1}{2}(-2u_0^3 + 2u^3)}{\sqrt{u_0^2 - u^2}^{\,3}}$$

$$= \pi + 2Mu_0 \int_0^1 dx \; \frac{1-x^3}{\sqrt{1-x^2}^{\,3}} = \pi + \delta\alpha(M)$$

- Integrate separately: $\displaystyle \frac{dx}{\sqrt{1-x^2}^{\,3}} = \frac{-\sin\theta \, d\theta}{\sin^3\theta} \qquad (x = \cos\theta)$

$$= -\frac{d\theta}{\sin^2\theta} = d(\cot\theta) = d\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\left[\text{recall: } \left(\frac{\cos}{\sin}\right)' = -1 - \frac{\cos^2}{\sin^2} = -\frac{1}{\sin^2}\right]$$

- Also: $\displaystyle \frac{-x^3 dx}{\sqrt{1-x^2}^{\,3}} = -\frac{1}{2}\frac{x^2 dx^2}{\sqrt{1-x^2}^{\,3}} = -\frac{1}{2}\frac{z\,dz}{\sqrt{1-z}^{\,3}} \qquad (x^2 = z = 1-y)$

$$= -\frac{1}{2}\frac{(1-y)(-dy)}{\sqrt{y}^{\,3}} = \frac{1}{2}\left(y^{-3/2}dy - y^{-1/2}dy\right)$$

$$= \frac{1}{2}\,d\left(-2y^{-1/2} - 2y^{1/2}\right) = d\left(-y^{-1/2}(1+y)\right)$$

$$= d\left(-\frac{2-x^2}{\sqrt{1-x^2}}\right)$$

- Combine both contributions: (We have to do this before inserting boundary values because of the divergence at $x \to 1$.)

$$\Rightarrow d\left(\frac{x-2+x^2}{\sqrt{1-x^2}}\right) = d\left(\frac{(1-x)(-2-x)}{\sqrt{1-x^2}}\right) = -d\left(\sqrt{\frac{1-x}{1+x}}\,(2+x)\right)$$

$$\Rightarrow \delta\alpha(M) = 2Mu_0\left[-\sqrt{\frac{1-x}{1+x}}\,(2+x)\right]_0^1 = 4Mu_0$$

Note: • $u_0 = 1/r_0 \approx 1/b$ at leading order in M.

- M appears together with G_N

$$\Rightarrow \underline{\underline{\delta\alpha = \frac{4MG_N}{b}}}$$

(For a light ray of a star, grazing the surface of the sun, one finds $\delta\alpha \approx 1.7''$ – this has been observed during a solar eclipse. Modern, higher precision measurements use radio waves of quasars.)