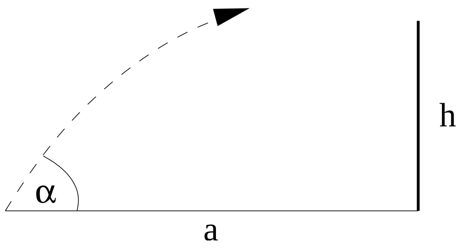
1. In-Class Exercises for Theoretical Physics I (Mechanics)

Solutions to the exercises: in the tutorials of the 2nd week (24th October, 2007)

Exercise 1:

A ball is thrown at an angle α above horizontal. At a distance a there is a wall of height h. What is the mimimum initial velocity v the ball has to have to get over the wall? Is there always a solution? (Neglect the air friction.)



Aufgabe 2:

Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. Show that

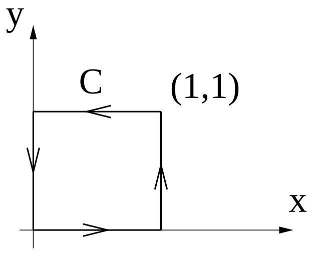
$$(a) \quad \nabla r = \hat{\mathbf{r}}, \qquad (b) \quad \nabla f(r) = \hat{\mathbf{r}}\,\frac{df}{dr}, \qquad (c) \quad \nabla^2 \ln r = \frac{1}{r^2}.$$

Exercise 3:

Let a vector function $\mathbf{F}$ be defined by

$$\mathbf{F} = (x+2)y^2\,\mathbf{e}_x + 4xy\,\mathbf{e}_y + 0 \cdot \mathbf{e}_z$$

and let C be a closed curve in the (x, y)-plane as shown in the figure.



Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{s}$ (a) directly and (b) using Stokes' theorem.

Exercise 4:

Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an oriented orthonormal basis of $\mathbb{R}^3$, such that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, etc. We define the Levi-Civita symbol ϵ_{ijk} via

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\, \mathbf{e}_k,$$

where repeated indices are to be summed over (Einstein summation convention).

a) Calculate the value of ϵ_{ijk} for arbitrary $i, j, k \in \{1, 2, 3\}$.

b) Express the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ using the ϵ-symbol and show the connection of the scalar triple product to the determinant.

c) Consider the double vector product $\mathbf{e}_i \times (\mathbf{e}_k \times \mathbf{e}_l)$ in order to show the important identities

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Hint: Determine first the coefficients a, b and c in the ansatz $\mathbf{e}_i \times (\mathbf{e}_k \times \mathbf{e}_l) = a\,\delta_{kl}\,\mathbf{e}_i + b\,\delta_{il}\,\mathbf{e}_k + c\,\delta_{ik}\,\mathbf{e}_l$.

d) Use the identity from c) to prove that

$$(\mathbf{a} \times \mathbf{b})^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

e) Show that

$$\epsilon_{ijk}\delta_{ij} = 0, \quad \epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}, \quad \epsilon_{ijk}\epsilon_{ijk} = 6.$$