12. Exercise Sheet for Theoretical Physics I (Classical Mechanics)

Solutions to be handed in in the tutorials of Week 13 (Jan. 30, 2008)

Exercise 43: (5 points)

a) Show that the following fundamental identities hold for the Poisson brackets of sets of mutually canonically conjugate variables:

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}.$$

b) Show moreover that the Poisson bracket of the angular momentum with the Hamiltonian vanishes for the two-dimensional simple harmonic oscillator with Hamiltonian

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(q_1^2 + q_2^2)$$

thus

$$\{L, H\} = 0.$$

Hint: L is the conserved quantity corresponding to the rotational symmetry of the problem.

Exercise 44: (5 points)

A particle of charge e and mass m moves in a magnetic field $\mathbf{B}(\mathbf{x})$ given by a vector potential $\mathbf{A}(\mathbf{x})$ via $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$. The appropriate Lagrangian is

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{e}{c}\,\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}).$$

a) Determine the canonical momenta p_i associated with the coordinates x_i and the Hamiltonian H (written as a function of these coordinates and momenta).

b) Determine $\mathbf{B}(\mathbf{x})$ for $\mathbf{A}(\mathbf{x}) = Bx_1\,\mathbf{e}_2$ with constant B.

c) Suppose that $\mathbf{A}$ is as given in part b). Set up Hamilton's equations and calculate the general solution for $\mathbf{x}(t)$. What shape is the trajectory? How can this result be physically interpreted?

Hint: The equations will look nicer using the abbreviation $\omega \equiv eB/mc$.

Exercise 45: (5 points)

a) Consider the following Lagrangian in one dimension (which is not the same as the Lagrangian of the simple harmonic oscillator!):

$$L = \frac{m\dot{q}^2}{2} + \frac{m\omega^2}{2}q^2.$$

Determine the Hamiltonian of the system and draw some representative trajectories (including the fixed point).

b) Consider the frictionless motion of a bead on a vertically standing (circular!) ring of wire in the gravitational field of the Earth. Contemplate the trajectories in the phase diagram qualitatively, and represent them graphically. Bear in mind the periodicity of suitably chosen coordinates and the occurrence of two fixed points.

c) Draw the trajectories for the problem of part b) in the case where the annulus is horizontal.

Exercise 46 (Additional exercise for interested people): (5 bonus points)

According to the Noether theorem there is a conserved quantity for every symmetry of a mechanical system. Conversely, conserved quantities may often be identified with infinitesimal symmetry transformations. We want to investigate this in more detail for the example of the Kepler problem.

Hint: The calculational effort in this exercise may be restricted to within reasonable limits by dextrously using the properties of the Poisson bracket. Use the canonical Poisson brackets $\{q_i, p_j\} = \delta_{ij}$, $\{p_i, p_j\} = \{q_i, q_j\} = 0$, and make use of the bilinearity, antisymmetry and the product rule (Leibniz' law) $\{fg, h\} = f\{g, h\} + \{f, h\}g$. The identity proved at the beginning of the semester $\epsilon_{ijk}\epsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}$ ist also helpful.

a) Show that the following equation holds for the components $L_i = \epsilon_{ijk}q_j p_k$ of the angular momentum $\mathbf{L}$:

$$\{L_i, p_j\} = \epsilon_{ijk}\, p_k, \qquad \{L_i, q_j\} = \epsilon_{ijk}\, q_k,$$

and thus show that

$$\{L_i, L_j\} = \epsilon_{ijk} L_k.$$

Commentary: It is already known that angular momentum is conserved in rotationally symmetric potentials. The rotation group in three dimensions SO(3) consists of 3×3 matrices $\mathbb{A}$ that may be written as $\mathbb{A} = \mathbb{1} + \theta_i \mathbb{L}_i + \mathcal{O}(|\boldsymbol{\theta}|^2)$ für ein $\boldsymbol{\theta} \in \mathbb{R}^3$ where the $\mathbb{L}_i$ $(i = 1, 2, 3)$ are in turn 3×3 matrices, the "generators" of $SO(3)$, and satisfy the following commutation relations[1]:

$$[\mathbb{L}_i, \mathbb{L}_j] = \epsilon_{ijk}\, \mathbb{L}_k.$$

[1] The commutator $[\mathbb{A}, \mathbb{B}]$ of two matrices $\mathbb{A}$ and $\mathbb{B}$ is defined as $[\mathbb{A}, \mathbb{B}] \equiv \mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}$; it is, like the Poisson bracket, bilinear, antisymmetric and is a derivation, thus also fulfilling the Jacobi identity.

The generators $\mathbb{L}_i$ of the rotation group thus represent the same algebra under the commutator as the components of angular momentum do under the Poisson bracket. In this sense, with the identification $\mathbb{L}_i \leftrightarrow L_i$, the symmetry "generates" the conserved quantities.

In the special case of the Kepler problem with potential $\sim 1/r$ the symmetry group is actually even larger than SO(3). There is namely one further conserved quantity, the Runge-Lenz vector $\mathbf{M}$ (*cf.* Ex. 23).
The Hamiltonian is

$$H(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}) \qquad \text{mit} \quad V(\mathbf{q}) = -\frac{\alpha}{|\mathbf{q}|}.$$

b) Show that for the components $M_i = \frac{1}{m}\epsilon_{ijk}\, p_j L_k + V(\mathbf{q})\, q_i$ of the Runge-Lenz vector the following equation holds:

$$\{L_i, M_j\} = \epsilon_{ijk}\, M_k.$$

Hint: Bear in mind that $\mathbf{L}$ is conserved. Thus, we have that $\{H, \mathbf{L}\} = 0$ and thus $\{V, \mathbf{L}\} = 0$ (convince yourself of this!).

In the same way one obtains that

$$\{M_i, M_j\} = -\frac{2H}{m}\, \epsilon_{ijk}\, L_k.$$

Now consider a bound state with known energy $H = E < 0$. Define

$$\mathbf{L}^{\pm} \equiv \frac{1}{2}\left(\mathbf{L} \pm \sqrt{-\frac{m}{2E}}\,\mathbf{M}\right).$$

c. Show by using the results above that

$$\{L_i^{\pm}, L_j^{\pm}\} = \epsilon_{ijk}\, L_k^{\pm}, \qquad \{L_i^{\pm}, L_j^{\mp}\} = 0.$$

Commentary: You have found two decoupled bases $\mathbf{L}^+$ und $\mathbf{L}^-$ of infinitesimal transformations that each represent the generating algebra of SO(3). The symmetry group is actually SO(4), whose infinitesimal structure is the same as that of SO(3)×SO(3). For scattering states with $H > 0$ the group SO(3,1) of Lorentz transformations results. This global structure of symmetry groups is not so easy to recognise. One can show [1] that the motion of a point mass in the Kepler potential can be mapped by a generalized canonical transformation onto the motion of a free point mass in a space of constant curvature. Negative (positive) energies then correspond to positive (negative) curvature.
Three-dimensional spaces of constant positive curvature are three-spheres with isometry group SO(4); 3-d. s. o. c. −ve c. are hyperbolic spaces with isometry group SO(3,1). In the special case $E = 0$ one obtains three-dimensional flat affine space with isometry group ISO(3) (translations und rotations). Prost!

Reference:

[1] A. Keane, R. Barrett und J. Simmons, "The classical Kepler problem and geodesic motion on spaces of constant curvature," *J. Math. Phys.* **41** (2000) 8108.