9. Exercise Sheet for Theoretical Physics I (Mechanics)

Solutions to be handed in in the tutorials of Week 12 (9th Jan. 2008)
Discussion of class exercise in Week 11 (19th Dec. 2007)

Exercise 34: (5 points)

The gravitational force resulting from a spherically symmetrical distribution of mass has exactly the same effect, in the space outside the sphere, as if the total mass were concentrated at the centre of the sphere.

a) Show that this is still correct for the so-called Yukawa force with the potential

$$V(r) = g\frac{e^{-\mu r}}{r}$$

except with an effective mass that is different from the total mass M of the sphere.

b) It is also true that the gravitational force in the interior of a hollow spherical shell (with spherically symmetric mass distribution) vanishes. Show that this is no longer the case for the Yukawa force with $\mu \neq 0$, but rather that a force is exerted on a test particle inside the sphere, that depends on the distance from the centre.

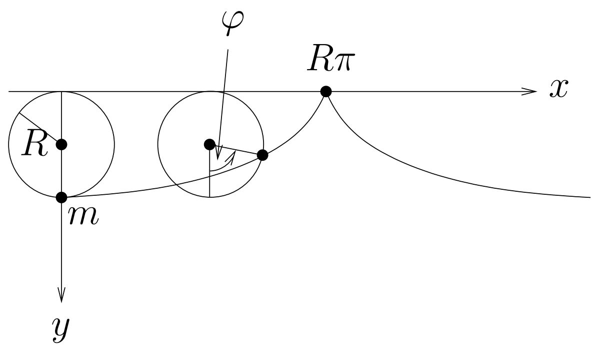
Exercise 35: (5 points)

A particle of mass m moves in the gravitational field of the earth upon a cycloid. The cycloid is implemented by rolling a circle of radius R on the underside of a (horizontal) straight line.

a) Consider the parametric representation of the cycloid thus constructed as a function of the angle φ (see diagram).

b) Take the angle φ as a generalised coordinate and determine the Lagrangian.

c) Set up the Euler-Lagrange equation and solve it. Give a physical interpretation of the result.



Exercise 36: (5 points)

Consider three point masses: $m_1 = 4m$ at $\mathbf{r}_1 = (a, 0, 0)$, $m_2 = m$ at $\mathbf{r}_2 = (0, 2a, 4a)$ and $m_3 = m$ at $\mathbf{r}_3 = (0, 4a, 2a)$, which are rigidly connected together (*e.g.* by massless rods).

a) Calculate the appropriate inertial tensor.

b) Determine the principal moments of inertia and the principal axes $\mathbf{e_i}$.

c) How large is the moment of inertia when the system rotates about an axis passing though the coordinate origin and the point $(x, y, z) = (0, 1, 1)$?

d) How large is the moment of inertia for a rotation about an axis parallel to the z-axis passing through the centre of gravity of the system?

Exercise 37: (Christmas Exercise) (5 bonus points)

At which height must little Konrad push a bauble (a hollow sphere with radius R) such that it rolls without sliding on a flat table? Calculate the moment of inertia explicitly. Assume that the initial impulse acts horizontally parallel to the table.

Exercise 33: (Class exercise) Brachistochrone

A slide begins at $x = 0$ and $y = 0$ and ends at $x = L$ and $y = -h$. Which shape must it be in order to minimise the time taken to slide all the way down?

Let the desired function whose graph specifies the form of the slide be $f(x)$.

a) Find an expression for the time as a functional

$$T[f] = \int_0^L \mathcal{L}(f(x), f'(x))\, dx.$$

Use conservation of energy.

b) Set up the Euler-Lagrange equation and simplify it. (Result: $1+f'^2+2ff'' = 0$.)

c) Show that this equation is equivalent to $ff'^2 + f - c = 0$ with c constant. Show further that, for $x = \frac{c}{2}(\theta - \sin(\theta))$ with arbitrary θ, a solution is given by $f(x) = \frac{c}{2}(1 - \cos(\theta))$. Convince yourself that this parameterises a cycloid (a cycloid is a curve described by one point of a circle that rolls upon a straight line).