Quantum Field Theory

(A. Hebecker ; Heidelberg ; WS 2015/16)

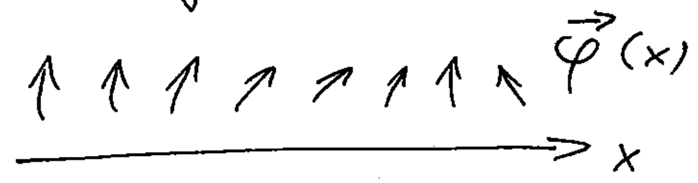
1 Introduction

1.1 Motivation

- You know from QM: The world is quantum at the fundamental level; classicality is only a particular limit
- Fields, in particular el. mag. fields (such as $\bar{E}(\bar{x},t)$; $\bar{B}(\bar{x},t)$; $A_\mu(\bar{x},t)$), are a fundamental part of reality $\Rightarrow$ We absolutely need a quantum theory of fields
- Obviously, the photon & its interactions will be one of the main practical applications
- less obvious (but see later): Also the electron will emerge as a quantum of an appropriate "electron field"
- In fact, all particles of the SM (Standard Model) are described by QFT. At sufficiently low energies (which is all we know at present), even the graviton fits perfectly into the framework of QFT.

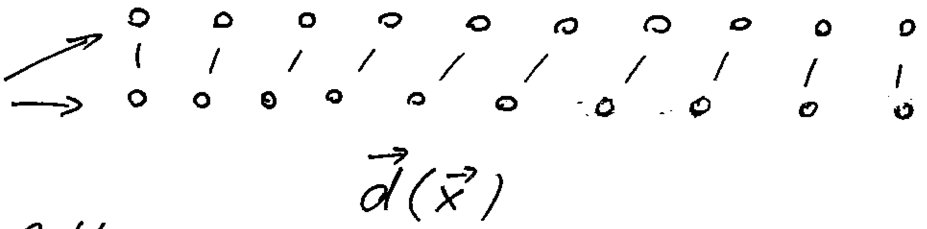
- Thus: QFT is the fundamental theory of this world (QM is its non-relativistic limit)
- QFT is also the most precisely tested theory we have
- Furthermore: "Effective" fields are central in Condensed Matter Theory (CMT):

(I) Spin field



(II) Displacement field ($\rightarrow$ phonons)

relative shift of atoms in lattice



- Most dynamics of these fields is in quantum regime $\Rightarrow$ needs QFT description $\Rightarrow$
- QFT is the modern language of CMT (including "cold atom physics"), even though it is just an effective description of the underlying many-body-QM

1.2 Symmetries (Poincare Invariance)

- As in <u>all</u> of physics, understanding the symmetries of the system we study will be essential
- Our stage is "space-time" $\mathbb{R}^4 \ni (t, \bar{x}) = \{x^\mu\}$

$\mu = 0, 1, 2, 3$

- To begin, focus only on space : $\bar{x} = \{x^i\}$

$i = 1, 2, 3$

<u>Symm:</u> Group of Transl. & Rotations

$$\bar{x} \longrightarrow \bar{x}' \quad \text{with} \quad x'^i = \underset{\substack{\uparrow \\ \text{rotation matrix}}}{R^i{}_j}\, x^j + d^i$$

- To simplify things further, focus just on rotations and ask which matrices R are "allowed".

- Clearly, we want lengths of vectors (e.g. distances) to be invariant :

$$|\bar{x}|^2 = \sum_{i=1}^{3} (x^i)^2 = x^i x^j \underset{\substack{\uparrow \\ \text{"Euclidean metric on } \mathbb{R}^3\text{"}}}{\delta_{ij}}$$

- Thus, $x'^i x'^j \delta_{ij} \stackrel{!}{=} x^i x^j \delta_{ij}$ or

$$\delta_{ij} R^i{}_k x^k R^j{}_l x^l = \delta_{ij} x^i x^j \quad \forall x .$$

- We need to demand

$$\delta_{ij} R^i{}_k R^j{}_l = \delta_{kl} \quad (R^T R = \mathbb{1}),$$

finding, as we of course knew, $R \in O(3)$.

<u>Summary</u>: space is $\mathbb{R}^3$ with eucl. metric (δ_{ij} in appropr. coord. system).
Symms. are transl. & "rots.", with the "rotations" defined by requiring invariance of the metric:

$$\delta_{ij} R^i{}_k R^j{}_l = \delta_{kl}$$

- The generalization to <u>space-time</u> and the <u>Poincaré group</u> is now straightforward:

$$\mathbb{R}^3 \ni \{x^i\} \longrightarrow \mathbb{R}^4 \ni \{x^\mu\} = \{t, x^i\}$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij} \longrightarrow \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$$

<u>Note</u>: The overall sign of the Minkowski-metric η is pure convention. One could replace

$\eta = \mathrm{diag}(1,-1,-1,-1)$ by $\eta = \mathrm{diag}(-1,1,1,1)$, which in fact makes more sense since the $\mathbb{R}^3$-part is unchanged. However, most QFT books use the "$+---$" convention.

Note: The relative sign between time & space part of η is a deep physical fact. If you wish, it is "observational data".

$$|\bar{x}|^2 = x^i x^j \delta_{ij} \longrightarrow x^2 = x^\mu x^\nu \eta_{\mu\nu} = t^2 - \bar{x}^2$$

$\boxed{\text{Note: } c \equiv 1 \text{ throughout!}}$ is the "invariant distance".

(It can be pos. $\to$ time-like separation
or neg. $\to$ space-like separation.)

$R \in O(3)$ if $\qquad\qquad \Lambda \in O(1,3)$ if

$$\delta_{ij} R^i{}_k R^j{}_l = \delta_{kl} \qquad\qquad \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

Poincare trfs: $\quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + d^\mu$

The Poincare group is the group of pairs (Λ, d) with $\Lambda \in O(1,3)$ & $d \in \mathbb{R}^4$ and with composition law

$$(\Lambda_1, d_1)\cdot(\Lambda_2, d_2) = (\Lambda_1\Lambda_2,\, \Lambda_1 d_2 + d_1).$$

- This will be our most important symm. group for this course. We will refer to $\mathbb{R}^4$ with this symm. as to $\mathbb{R}^{1,3}$ or Minkowski-space.

- Clearly, rotations are a subgroup of Lorentz-trfs.:

$$\Lambda = \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & R \end{array}\right) \text{ with } R \in SO(3) \text{ is a rotation in } O(1,3)$$

- The subgroup of "special Lorentz trfs." is defined by demanding $\det(\Lambda) = 1$ and $\Lambda^0{}_0 > 0$. It is also the identity component $SO^+(1,3) \subset O(1,3)$.

- For $\mathbb{R}^{1,1}$, Λ is obviously just a 2×2 matrix and we can be very explicit:

$$\Lambda = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \in SO^+(1,1)$$

is the general group element.

- $$\begin{pmatrix} t \\ x \end{pmatrix} \longrightarrow \Lambda \cdot \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t\cosh\alpha + x\sinh\alpha \\ x\cosh\alpha + t\sinh\alpha \end{pmatrix}$$

is obviously a boost with $\beta = v/c = v$

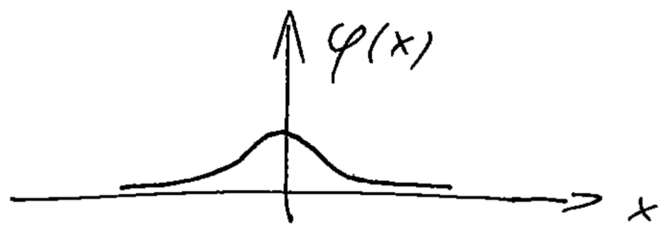
$$\cosh\alpha = \frac{1}{\sqrt{1-\beta^2}} \quad ; \quad \sinh\alpha = \frac{\beta}{\sqrt{1-\beta^2}} .$$

1.3 Symmetries acting on fields

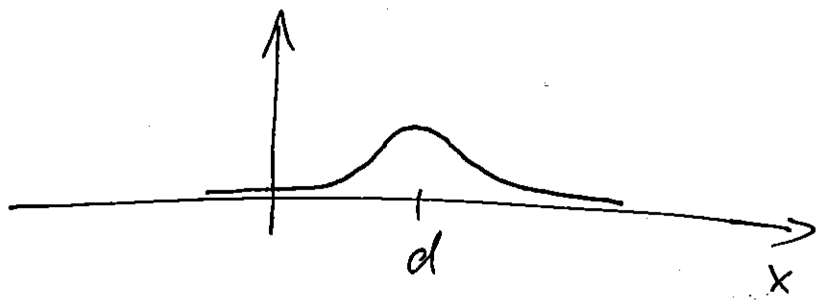
- Consider some scalar field configuration, including its (classical) evolution in time. Mathematically, this is given by a fct.

$$\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R} \; ; \; \{x^\mu\} \equiv x \longmapsto \varphi(x).$$

- For simplicity, let's first replace $\mathbb{R}^4$ by $\mathbb{R}$ (e.g. for a 1-dim. world at fixed time). Consider a field configuration "localized" near 0:



Perform a translation by d ("active point of view", not coord. trf. !)



- Clearly, we need to demand that for

$$x \xrightarrow{d} x' = x + d \; ; \; \varphi \xrightarrow{d} \varphi'$$

we get $\varphi'(x') = \varphi(x)$

or $\varphi'(x+d) = \varphi(x)$

or $\varphi'(x) = \varphi(x-d)$

- This makes sense: If φ had its maximum at $x=0$, φ' should have its max. at $x=d$. Thus, φ' is defined by applying the <u>inverse</u> symm. operation to the argument.

- General case: ($\mathbb{R}^{1,3}$; Poinc. group)

For $\varphi \xrightarrow{(\Lambda,d)} \varphi'$, we have:

$$\varphi'(x') = \varphi(x)$$
$$\varphi'(\Lambda x + d) = \varphi(x) \qquad \leftarrow \{x \to \Lambda^{-1}x\}$$
$$\varphi'(x+d) = \varphi(\Lambda^{-1}x) \qquad \leftarrow \{x \to x-d\}$$
$$\boxed{\varphi'(x) = \varphi(\Lambda^{-1}\cdot(x-d))}$$

- Let us also look at the trf. of the vector field $\{\partial_\mu \varphi\} \equiv \{\frac{\partial}{\partial x^\mu}\varphi\} = \{\frac{\partial}{\partial x^0}\varphi(x^0, \dots, x^3), \dots, \dots, \dots\}$.

(For simplicity, let $d=0$ here.)

$$\partial_\mu \varphi' = \frac{\partial}{\partial x^\mu}\varphi(y(x)) \quad ; \quad y = \Lambda^{-1}\cdot x$$

$$= \frac{\partial y^\nu}{\partial x^\mu}\cdot\frac{\partial}{\partial y^\nu}\varphi(y) = \frac{\partial}{\partial x^\mu}((\Lambda^{-1})^\nu{}_\rho x^\rho)\,\partial_\nu\varphi(y)$$

$$= (\Lambda^{-1})^\nu{}_\mu(\partial_\nu\varphi)(\Lambda^{-1}x)$$

<u>Simple math. excursion:</u> Dual vector space ...
... inverse metric

Let x^μ, $\mu = 0 \ldots 3$ be an element of $\mathbb{R}^{1,3} \equiv V$.
Let the elements of V^* be given by y_μ, $\mu = 0 \ldots 3$ such that $x \cdot y \equiv x^\mu y_\mu$. The metric η provides a natural map

$$V \longrightarrow V^* \; ; \; x^\mu \longmapsto \eta_{\mu\nu} x^\nu .$$

This map & its inverse are frequently referred to as "lowering/raising indices" with $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

$$\left(\text{Note: } \eta_{\mu\nu} \eta^{\nu\rho} = \delta_\mu{}^\rho \equiv \eta_\mu{}^\rho . \right)$$

Now we know that $\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$

$$\Rightarrow \quad \Lambda^T \eta \Lambda = \eta$$

$$\Rightarrow \quad \eta^{-1} \Lambda^T \eta = \Lambda^{-1}$$

$$\Rightarrow (\Lambda^{-1})^\mu{}_\nu = \eta^{\mu\rho} (\Lambda^T)_\rho{}^\sigma \eta_{\sigma\nu} = \eta^{\mu\rho} \Lambda^\sigma{}_\rho \eta_{\sigma\nu}$$

$\boxed{\text{$\Lambda$ with lowered/raised indices}} \longrightarrow \; \equiv \Lambda_\nu{}^\mu$

Thus: $$\partial_\mu \varphi'(x) = \underbrace{\Lambda_\mu{}^\nu}_{} \partial_\nu \varphi(\Lambda^{-1} x)$$

Derivative transforms as element of V^*.
(like $x_\mu \longrightarrow \Lambda_\mu{}^\nu x_\nu$).