10 Quantization of Spinors

10.1 The Hamiltonian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,; \qquad \psi = \{\psi_a\}$$

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = \frac{\partial}{\partial \dot{\psi}_a}(i\psi^+\gamma^0\gamma^0\dot{\psi}) = i\psi^{+a} \implies \underline{\underline{\pi = i\psi^+}}$$

(with π interpreted as row-vector)

- We see that the lagrangian (and hence the hamiltonian) can be given using just ψ & $\pi \sim \psi^+$. There is no need to introduce a further canonical momentum belonging to ψ^+. This different from the complex scalar, where both ϕ, π & ϕ^+, π^+ where really needed. The reason is that the EOMs are only 1st order in t, hence one has fewer initial conditions & fewer d.o.f.

... cf. Weinberg, Sect. 7 for more details.

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^+\dot{\psi} - \psi^+\gamma^0(i\partial\!\!\!/ - m)\psi = -\bar{\psi}(i\gamma^i\partial_i - m)\psi$$

$$\underline{\underline{= i\pi\gamma^0(i\gamma^i\partial_i - m)\psi}}$$

10.2 Quantization attempt with commutators

$$[\psi(\bar{x}), \pi(\bar{y})] = [\psi(\bar{x}), i\psi^+(\bar{y})] = i\delta^3(\bar{x}-\bar{y})\cdot\mathbb{1}$$

- Skipping familiar steps, jump directly to ansatz for field in terms of creation/annih. operators multiplying the known classical solutions (as for compl. scalar):

$$\psi(x) = \int d\tilde{p}\,\left(a^s_{\bar{p}}\, u_s(p)\, e^{-ipx} + b^{s+}_{\bar{p}}\, v_s(p)\, e^{ipx}\right)$$

($\sum_s$ implied)

where $[a^r_{\bar p}, a^{s+}_{\bar q}] = (2\pi)^3 \delta^3(\bar p - \bar q)\, \delta^{rs}\, 2p_0$ & same for b.

- We must check consistency with our original commut. relations: (at equal time, i.e. $x^0 = y^0$)

$$[\psi(\bar x), \psi^+(\bar y)] = \int d\tilde p\, d\tilde q \Big(e^{i\bar p\bar x - i\bar q\bar y} u_s(p) u_r^+(q) [a^s_{\bar p}, a^{r+}_{\bar q}]$$
$$+ e^{-i\bar p\bar x + i\bar q\bar y} v_s(p) v_r^+(q) [b^{s+}_{\bar p}, b^r_{\bar q}] \Big)$$

$$= \int d\tilde p \Big[e^{i\bar p(\bar x - \bar y)} (\not p + m) - \underbrace{e^{-i\bar p(\bar x-\bar y)}(\not p - m)}_{\substack{\text{change } \bar p \to -\bar p \text{ appealing to symm.} \\ \text{of the integration measure.}}} \Big] \cdot \gamma^0$$

$$= \int d\tilde p\, e^{i\bar p(\bar x - \bar y)} \big(p^0\gamma_0 + p^i\gamma_i + m - (p^0\gamma_0 - p^i\gamma_i - m)\big)\gamma^0.$$

- We see this will fail since the p^0-term, needed to cancel $1/p^0$ from $d\tilde p$, will drop out. However, we can try to assume $[b, b^+] = -1$, effectively exchanging the roles of b & b^+. This appears to work:

$$\ldots = \int d\tilde p\, e^{i\bar p(\bar x - \bar y)}\, 2p^0\gamma_0 \cdot \gamma^0 = \delta^3(\bar x - \bar y) \cdot \mathbb{1}.$$

- However, we finally fail when it comes to H. A straight-forward calculation (for details see a similar calculation below) gives:

$$H = \int d\tilde p\, p_0 \sum_s \big(a^{s+}_{\bar p} a^s_{\bar p} - b^{s+}_{\bar p} b^s_{\bar p} \big)$$

This wrong sign (independent of $b^+ \leftrightarrow b$) makes the energy unbounded below and hence the vacuum unstable. No "easy" cure is known!

10.3 Quantization with anticommutators

- The only known cure to the problem above is to <u>fundamentally</u> change the rules of the game and to <u>quantize with anticommutators.</u>

[It can be proven rigorously that this must be done for all fields with half-integer spin. This is known as the "Spin-Statistics-Theorem", see e.g. Streater/Wightman: "PCT, Spin and Statistics, and All That".]

- At the calculational level, we just repeat Sect. 10.2, but with the <u>new postulate</u>:

$$\{\psi(\bar{x}), \pi(\bar{y})\} = \{\psi(\bar{x}), i\psi^+(\bar{y})\} = i\delta^3(\bar{x}-\bar{y})$$

$$\Updownarrow$$

$$\{a^r_{\bar{p}}, a^{s+}_{\bar{q}}\} = \{b^r_{\bar{p}}, b^{s+}_{\bar{q}}\} = (2\pi)^3 2p_0\, \delta^3(\bar{p}-\bar{q})\, \delta^{rs}$$

& all other anticomm.s $= 0$.

- Again, as in Sect. 10.2 we find

$$H = \int d\tilde{p}\, p_0 \sum_s \left(a^{s+}_{\bar{p}} a^s_{\bar{p}} - b^s_{\bar{p}} b^{s+}_{\bar{p}}\right)$$

$$H = \int d\tilde{p}\, p_0 \sum_s \left(a^{s+}_{\bar{p}} a^s_{\bar{p}} + b^{s+}_{\bar{p}} b^s_{\bar{p}}\right) + \underbrace{(\sim \mathbb{1})}_{\text{dropped, as in scalar case}}$$

This is where the sign-problem of 10.2 is solved due to $AB = -BA + \ldots$

- Now that the conceptual idea is clear, let's do the actual calculation giving H in some detail:
- Use $H = \int d^3x\, \bar{\psi}\,(-i\bar{\gamma}\bar{\nabla} + m)\,\psi$

& $\psi(\bar{x}) = \int d\tilde{p}\, \left(a^s_{\bar{p}} u_s(p) e^{i\bar{p}\bar{x}} + b^{s+}_{\bar{p}} v_s(p) e^{-i\bar{p}\bar{x}}\right)$

$\bar{\psi}(\bar{x}) = \int d\tilde{p}'\, \left(a^{s'+}_{\bar{p}'} \bar{u}_{s'}(p') e^{-i\bar{p}'\bar{x}} + b^{s'}_{\bar{p}'} \bar{v}_{s'}(p') e^{i\bar{p}'\bar{x}}\right)$

& $\int d^3\bar{x}\, e^{i\bar{p}\bar{x} \pm i\bar{p}'\bar{x}} = (2\pi)^3 \delta^3(\bar{p} \pm \bar{p}')$.

- We find terms involving a^+a, a^+b^+, ba, bb^+.

① $H_{a^+a} = \int \frac{d\tilde{p}}{2p^0}\, a_{\bar{p}}^{s'+} a_{\bar{p}}^{s}\, \bar{u}_{s'}(p)(\bar{\gamma}\bar{p}+m)u_s(p)$

Use $0 = (\not{p}-m)u(p) = (\gamma^0 p^0 - \bar{\gamma}\bar{p} - m)u(p)$ [Note: $\bar{p} = \{p^i\}$, $\bar{\nabla} = \{\frac{\partial}{\partial x^i}\}$]

$\Rightarrow (\bar{\gamma}\bar{p}+m)u(p) = \gamma^0 p^0 u(p)$

$\Rightarrow H_{a^+a} = \int \frac{d\tilde{p}}{2}\, a_{\bar{p}}^{s'+} a_{\bar{p}}^{s}\, \bar{u}_{s'}(p)\gamma^0 u_s(p)$

Useful relations: $\boxed{\bar{u}_r(p)\gamma^0 u_s(p) = \bar{v}_r(p)\gamma^0 v_s(p) = 2p_0\,\delta_{rs}}$

Derivation: We know: $\boxed{(\not{p}-m)u(p) = 0.}$

It follows: $0 = u(p)^+(\not{p}^+ - m) = u(p)^+(\not{p}^+ - m)\gamma^0 = \bar{u}(p)(\not{p}-m)$

$\Rightarrow \boxed{\bar{u}(p)(\not{p}-m) = 0}$

Using these, we have: $\bar{u}_r(p)\gamma^0 u_s(p) = \frac{1}{2m}\bar{u}_r(p)\{m,\gamma^0\}u_s(p)$

$= \frac{1}{2m}\bar{u}_r(p)\{\not{p}-m+m,\gamma^0\}u_s(p) = \frac{1}{2m}\bar{u}_r(p)\{\not{p},\gamma^0\}u_s(p)$

$= \frac{p_0}{m}\bar{u}_r(p)u_s(p) = \frac{p_0}{m}2m\delta_{rs} = 2p_0\,\delta_{rs}$.

(The relation with v is derived analogously.)

$\Rightarrow H_{a^+a} = \int d\tilde{p}\, p_0\, a_{\bar{p}}^{s+} a_{\bar{p}}^{s}$

②, ③ $H_{a^+b^+} = H_{ba} = 0$ since $u_s^+(p^0,-\bar{p})v_r(p^0,\bar{p})$

$= v_s^+(p^0,-\bar{p})u_r(p^0,\bar{p}) = 0$

Prove this!

Two useful relations!

(4) $$H_{bb^+} = \int \frac{d\tilde{p}}{2p_0}\, b_{\bar p}^{s'} b_{\bar p}^{s+} \underbrace{\bar{v}_{s'}(p)(-\vec{\gamma}\vec{p}+m)v_s(p)}_{\bar{v}_{s'}(p)(-\gamma^0 p^0)v_s(p)}$$

in analogy to earlier argument

$$\Rightarrow H_{bb^+} = \int d\tilde{p}\, p_0 \left(-b_{\bar p}^{s} b_{\bar p}^{s+}\right).$$

Thus, as claimed before

$$\underline{\underline{H = \int d\tilde{p}\, p_0 \left(a_{\bar p}^{s+} a_{\bar p}^{s} + b_{\bar p}^{s+} b_{\bar p}^{s}\right)}} + \text{ irrelev. const.}$$

Define Fock space as in bosonic case:

$$|0\rangle;\quad a_{\bar p}^{s+}|0\rangle,\ b_{\bar p}^{s+}|0\rangle;\quad a_{\bar p}^{s+} a_{\bar p}^{r+}|0\rangle,\ \dots ,$$

where as before

$$a_{\bar p}^{s}|0\rangle = b_{\bar p}^{s}|0\rangle = 0 \quad \forall\, \bar p, s.$$

<u>Crucial difference</u>: Since we used <u>anti</u>-commut. rels., $(a_{\bar p}^{s+})^2|0\rangle = 0$, i.e., multiparticle states where two particles have equal labels <u>never</u> occur. Our particles are <u>fermions</u>. (Hence the name "spin-<u>statistics</u>-theorem")

[Actually, for plane waves $\bar p = \bar p'$ "never occurs" anyway. To define the above more properly one should go to e.g. to a finite volume, where $\bar p$ is discrete and $(a_{\bar p}^{+})^2 = 0$ makes sense in a more straightforward way.]

<u>10.4 Time ordering, Greens fcts., Dirac propagator</u>

- Basic object of interest:

$$\langle 0|T(\text{product of } \psi\text{'s and } \bar\psi\text{'s})|0\rangle$$

- Due to the anticommut. rels., the definition of T is changed in a crucial way:

$$T\,\psi_{a_1}(x_1)\cdots\psi_{a_n}(x_n) \equiv \mathrm{sgn}(p)\,\psi_{a_{p(1)}}(x_{p(1)})\cdots\psi_{a_{p(n)}}(x_{p(n)})$$

where $\{p(1),\dots,p(n)\}$ is a permutation "p" of $\{1,\dots,n\}$ such that $x^0_{p(1)} \geq \dots \geq x^0_{p(n)}$.

$\mathrm{sgn}(p) = \pm 1$ for even/odd p. [Same for $\bar\psi$'s or mix of ψ's & $\bar\psi$'s.]

- With this definition, the LSZ-formula still holds (up to a possible overall minus-sign, which we here ignore)

- The relation between time ordered Greens fcts. for interacting & free fields also still holds (recall that, crucially, this is where $\exp(iS_{int})$ enters).

- Finally, in the last step towards Feynman rules, the Wick-theorem is crucially affected. (In fact, without the above adjustment in the definition of T we would not be able to derive a Wick-theorem at all.

Wick-thm. for spinor fields

$$T(\Pi\,\psi_i\bar\psi_j) = :(\Pi\,\psi_i\bar\psi_j + \text{all poss. contractions}):$$

Here, by definition, a "contraction" includes a factor (-1) for each exchange of neighboring $\psi,\bar\psi$'s required to put contracted pairs next to each other.

Example: $:\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4: \; = - \psi_1\bar{\psi}_3 :\psi_2\bar{\psi}_4:$

- A contraction is defined exactly as in the bosonic case:

$$\psi_a(x)\bar{\psi}^b(y) \equiv \langle T\psi_a(x)\bar{\psi}^b(y)\rangle \equiv S_F(x-y)_a{}^b .$$

- Contractions $\psi\psi$ & $\bar{\psi}\bar{\psi}$ vanish or, if you wish, don't exist.
- The "Dirac propagator" (the index F still stands for the "Feynman-$i\varepsilon$-prescription") reads

$$S_F(x-y)_a{}^b = \int \frac{d^4p}{(2\pi)^4}\cdot\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\cdot e^{-ip(x-y)}$$

(The derivation is as in the bosonic case, just with a bit more algebra to obtain the "$\not{p}+m$" from the u's & v's of the field decomposition.) $\to$ Problems.

- An alternative "derivation" is as follows: (The fact that this is a proper derivation will only become clear in the path integral approach.)

  - For the scalar, we have

$$-(\Box_x+m^2)D(x-y) \equiv i\delta^4(x-y) \quad \text{in general}$$

(this makes D a "Greens fct.")

and, in particular,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\cdot\frac{i}{p^2-m^2+i\varepsilon}\cdot e^{-ip(x-y)}$$

This denominator is obviously just "$-(\Box+m^2)$" in Fourier space.

(Depending on the pole-prescription we get Feynman, retarded or advanced Green's fcts.)

- Analogously

$$(i\not{\partial}_x - m)\, S(x-y) = \mathbb{1}\cdot i\delta^4(x-y)$$

with

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\cdot \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\cdot e^{-ip(x-y)}.$$

- The crucial piece of algebra to remember is

$$(i\not{\partial}_x - m) \longrightarrow (\not{p}-m) \quad \& \quad \frac{(\not{p}-m)(\not{p}+m)}{p^2-m^2} = \frac{p^2-m^2}{p^2-m^2} = \mathbb{1}$$

or, equivalently, $\displaystyle \frac{\not{p}+m}{p^2-m^2} = \frac{1}{\not{p}-m}$.

10.5 U(1)-Symmetry of the Dirac Lagrangian

$\mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi$; $\psi \to e^{-i\varepsilon}\psi$ is a global symmetry.

- Recall the Noether theorem:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} X - F^\mu \text{ is conserved, where}$$

1) the symm. trf. is $\varphi \to \varphi + \varepsilon X$

2) $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu F^\mu$

- Here ($F=0$; $X = -i\psi$) we find

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}(-i\psi) = \bar{\psi}i\gamma^\mu(-i\psi) = \underline{\underline{\bar{\psi}\gamma^\mu\psi}}.$$

(This will become the electromag. current!)

and

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi^+\psi = \underline{\underline{\int d\tilde{p} \sum_s \left(a_{\bar{p}}^{s+}a_{\bar{p}}^{s} - b_{\bar{p}}^{s+}b_{\bar{p}}^{s}\right)}}$$

10.6 Yukawa theory

- The arguably simplest interacting theory with fermions is the Yukawa theory:

$$\mathcal{L} = \underbrace{\bar{\psi}(i\not{\partial} - m)\psi}_{\text{free}} + \underbrace{\frac{1}{2}(\partial\varphi)^2 - \frac{\mu^2}{2}\varphi^2}_{\text{free}} + \underbrace{\lambda\varphi\bar{\psi}\psi}_{\text{interaction}}$$

- It plays e.g. a (phenomenological) role in nuclear interactions and a (as far as we know fundamental) role in the fermion-mass generation in the SM.

- For more details see "Christmas problem"...

- We now turn to the more complicated and structurally more interesting gauge interactions of fermions.