11 Quantum Electrodynamics

11.1 Lagrangian

The logic is precisely the same as in our discussion of scalar QED:

- promote the global $U(1)$-symm. of $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ to a local symm: $\psi \to e^{-i\alpha(x)}\psi$.
- to maintain invariance we need to also promote ∂_μ to $D_\mu = \partial_\mu + iA_\mu$
- we also must add a kin. term for A_μ.

Hence:

$$\mathcal{L}_{QED} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!/ - m)\psi$$

$$[D\!\!\!/ \equiv \gamma^\mu D_\mu]$$

or

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!/ - m)\psi \quad \text{with } D_\mu = \partial_\mu + ieA_\mu}$$

(Check of invariance (as in scalar case):

$$D_\mu\psi \to D'_\mu\psi' = (\partial_\mu + ieA'_\mu)e^{-ie\alpha(x)}\psi = \ldots$$

$$\ldots = e^{-ie\alpha(x)}D_\mu\psi \quad \text{if } A'_\mu = A_\mu + \partial_\mu\alpha)$$

11.2 Feynman rules

Write $\mathcal{L}_{QED} = \mathcal{L}_{free} + \mathcal{L}_{int}$ with

$$\mathcal{L}_{free} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \& \quad \mathcal{L}_{int.} = \underline{\underline{-e\bar{\psi}A\!\!\!/\psi}}$$

(so-called "minimal coupling" needed for gauge inv.)

Side remark: A "non-minimal" coupling would e.g. be $\mathcal{L}_{int} = \frac{1}{\Lambda} F_{\mu\nu} \bar{\psi} \gamma^\mu \gamma^\nu \psi$. It would make the theory "non-renormalizable" (see later) and it is also less important if Λ is large. Note:

$[\varphi] = [A_\mu] = [\partial_\mu] = E^1$; $[\psi] = E^{3/2}$

"mass" or "energy" dimension of..

since $[S] = 1$; $[d^4x] = E^{-4}$; $[\mathcal{L}] = E^4$.

Thus $[\Lambda] = E$; it typically is the energy scale at which some "new physics" appears to "generate" the above non-minimal coupling.*

(* also: "higher-dimension operator" in this case $[F\bar{\psi}\psi] = E^5$)

$\mathcal{L}_{free}$ implies:

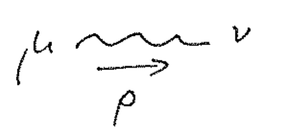
$$\underset{a \quad\; p \quad\; b}{\longrightarrow} = \underbrace{\left(\frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon} \right)_a^{\;b}}$$

as before, this is just the Fourier-space expression for $\langle T \psi_a(x) \bar{\psi}^b(y) \rangle$.

Comment: We also have

$$\frac{i}{\not{p}-m+i\varepsilon} = \frac{i(\not{p}+m-i\varepsilon)}{(\not{p}+m-i\varepsilon)(\not{p}-m+i\varepsilon)} = \frac{i(\not{p}+m-i\varepsilon)}{p^2-(m-i\varepsilon)^2}$$

$$= \frac{i(\not{p}+m-i\varepsilon)}{p^2-m^2+2mi\varepsilon+\varepsilon^2} \mathrel{\hat{=}} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} \qquad (\varepsilon \to \varepsilon')$$



$= \frac{-i\eta^{\mu\nu}}{p^2+i\varepsilon}$ (in our simplest gauge choice, which is also known as "Feynman gauge")

$\underline{\mathcal{L}_{int.}\text{ implies:}}$

$= ie(\gamma_\mu)_b{}^a$

(The structure of this expression should be obvious: it is just the coefficient of the "3-field-term" in $\mathcal{L}$. The sign needs some care...)

- We derive the last Feynman rule ("the vertex") using the "imagined" process $e^+ + \gamma \to e^+$:
(momenta: $p + k = p'$)

$$\langle 0 | a^{s'}_{\bar{p}'} \left(i \int d^4x\, \mathcal{L}_{int}\right) a^{s+}_{\bar{p}} a^{\mu+}_{\bar{k}} \varepsilon_\mu(k) | 0 \rangle = (2\pi)^4 \delta^4(\ldots) i\mathcal{M}_{fi}$$

outgoing e^+ with spin s' — incoming e^+ with spin s and photon with polariz. ε_μ.

Thus:

$$\langle 0 | a^{s'}_{\bar{p}'} \left[(-ie) \int d^4x\, \bar{\psi}(x) \gamma_\nu A^\nu(x) \psi(x)\right] a^{s+}_{\bar{p}} a^{\mu+}_{\bar{k}} | 0 \rangle\, \varepsilon_\mu(k)$$

Recall:

$$\psi(x) = \int d\tilde{q}\, a^r_{\bar{q}} u_r(q) e^{-iqx} + \cdots$$

$$\bar{\psi}(x) = \int d\tilde{q}\, a^{r+}_{\bar{q}} \bar{u}_r(q) e^{iqx} + \cdots$$

$$A^\nu(x) = \int d\tilde{q}\, a_{\bar{q}}^\nu\, e^{-iqx} + \cdots$$

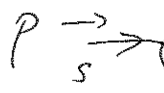
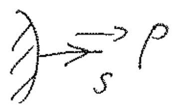
$$\{a^r_{\bar{q}}, a^{s+}_{\bar{p}}\} = 2p_0 (2\pi)^3 \delta^3(\bar{p}-\bar{q}) \delta^{rs}$$

$$[a_{\bar{q}}{}^\nu, a^{+\mu}_{\bar{k}}] = -2k_0 (2\pi)^3 \delta^3(\bar{k}-\bar{q}) \eta^{\nu\mu}$$

- After a small orgy of integration, we find

$$i\mathcal{M}_{fi} = \underbrace{\bar{u}_{s'}(p')}_{\text{outg. state}} \underbrace{(ie\gamma^\mu)}_{\substack{\text{vertex} \\ \text{(as given above)}}} \underbrace{u_s(p)\,\varepsilon_\mu(k)}_{\text{incom. state}}$$

- Thus, we confirm the vertex-Feynm. rule given above together with:

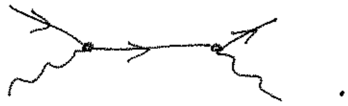
 $= u_s(p)_a$;  $= \bar{u}_s(p)^a$

incoming positron — outgoing positron

 $= \varepsilon_\mu(k)$ (same with "*" for outgoing photon)

incoming photon.

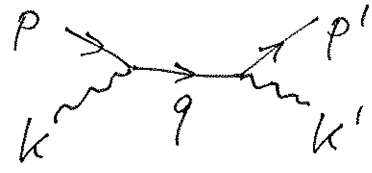
Connection of vertex to propagator

- We have only considered a situation where the vertex connects to ext. lines – we also need to understand how it connects to propagators
- To do so, consider $e^+\gamma \to e^+\gamma$;  .
- The corresp. amplitude follows from:

$$\langle 0|a^s a^\mu T\Big(\int_x \bar{\psi}(-ie\gamma_\nu A^\nu)\psi\Big)\Big(\int_y \bar{\psi}(-ie\gamma_\beta A^\beta)\psi\Big) a^{s+} a^{\mu +}|0\rangle$$

[Here we used the symbol "⌐‑‑‑¬" to say that the corresponding a/a^+'s produce a non-zero number by commut. relations – in analogy to what happens in a proper "contraction".]

- Working this out in detail one finds the (more or less obvious) result:



$$= \varepsilon^*_\mu(k')\bar{u}_{s'}(p')(ie\gamma^\mu)\frac{i}{\not{q}-m+i\varepsilon}(ie\gamma^\nu)u_s(p)\,\varepsilon_\nu(k)$$

to be read from left to right

to be read from right to left (we use matrix notation and contract spinor indices as we go along the line "$\rightarrow$")

- Important conclusion: We have learned that our def. of the arrow on the propagator (going from $\bar\psi$ to ψ) is consistent with the way in which we introduced it in external particles ($\rightarrow$ ← for incoming positron)

- Finally, to describe external antiparticles (electrons) consider:

$$e^-\gamma \rightarrow e^- \quad (\text{with } p+k=p')$$

$$\Rightarrow \quad \langle 0|\, b_{s'}\left(\int \bar\psi(-ie\not{A})\psi\right) b_s^+\, a^{\mu+}|0\rangle$$

$\Rightarrow$

$$= \varepsilon^*_\mu(k)\,\bar{v}_s(p)\,(ie\gamma^\mu)\,v_{s'}(p')$$

time flows left to right $\longrightarrow$ this translates, in contrast to the particle case, into time flowing again left to right.

- Summary: In the matrix notation for a fermion line, time flows right $\rightarrow$ left / left $\rightarrow$ right for particle / antiparticle.

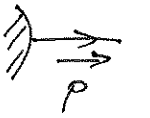
Summary of QED Feynman rules:

 $= i/(\not{k}-m+i\varepsilon)$

 $= -i\eta^{\mu\nu}/(k^2+i\varepsilon)$

 $= ie\gamma^\mu$

 $= (\cdots)\,u(p)$ [incom. part.]

 $= \bar{u}(p)(\cdots)$ [outg. part.]

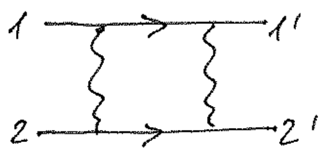
 $= \bar{v}(p)(\cdots)$ [incom. antipart.]

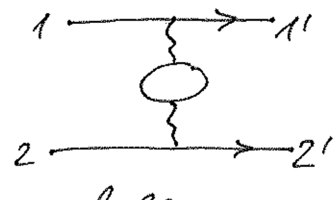
 $= (\cdots)\,v(p)$ [outg. anti-part.]

Additional rules: A diagram receives a relative minus-sign for ① every closed fermion loop
② the exchange of two external fermion lines (relative to another diagram).

- For reasons of time, we limit ourselves to a brief illustration of the logic behind the two last rules.

①: Consider the following two diagrams contributing to $e^+e^+ \to e^+e^+$ at order e^4 for the amplitude ("NLO"):





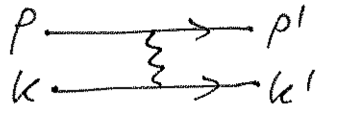
The underlying contractions are as follows:

$$\langle 2'1'(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)12\rangle \qquad \langle 2'1'(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)12\rangle$$

Each diagram acquires two minus-signs "from intersecting fermion-contraction-lines". In addition, the r.h. diagram has a contraction $\bar\psi\psi = -\psi\bar\psi = -S_F$. Such an extra minus always arises when there is closed fermion loop.

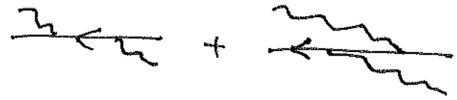
②: Here, the minus comes from $\langle 0|a_{s'}a_s = -\langle 0|a_s a_{s'}$.

An example is given by



vs.

11.3 Elementary processes

- Compton scattering: $e^-\gamma \to e^-\gamma$; 

  (Result: Klein-Nishina formula)

- Elastic e^-e^- (or e^+e^+)-scattering: $e^-e^- \to e^-e^-$;

  (also: Møller scattering)

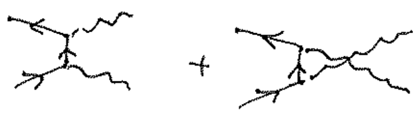
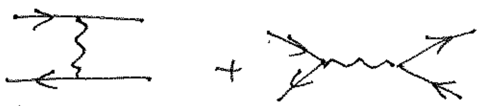
Special case: non-relativistic (heavy) target + highly-relativistic projectile (e.g. e^+ off μ^+ or nucleus)

$\Rightarrow$ "Coulomb scattering" (Result: Mott formula)

name implies scattering off static Coulomb field

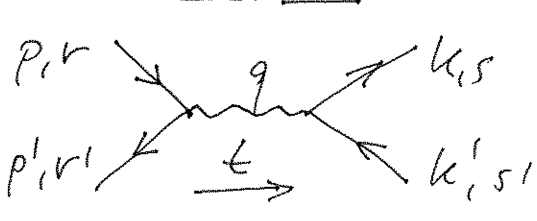
Special case: both projectile & target non-relativistic

$\Rightarrow$ "Rutherford scattering" (Result: Rutherford formula)

- Pair-annihilation to photons : $e^+e^- \to \gamma\gamma$; 
- Bhaba-scattering : $e^+e^- \to e^+e^-$; 
- Light-by-light scattering : $\gamma\gamma \to \gamma\gamma$; 

  (no "tree-level" process exists ; at least 1-loop required)

Our example: $e^+e^- \to \mu^+\mu^-$ (even simpler than Bhaba-scatt., since only one diagram)



$$q = p + p' = k + k' \quad ; \quad q^2 \equiv s$$

$$i\mathcal{M} = \bar{u}_s(k)\, ie\gamma_\mu v_{s'}(k') \cdot \left(\frac{-i\eta^{\mu\nu}}{q^2}\right) \cdot \bar{v}_{r'}(p')\, ie\gamma_\nu u_r(p)$$

$$d\sigma = \frac{1}{2s}|\mathcal{M}|^2 dX^{(2)} = \frac{1}{64\pi^2 s}|\mathcal{M}|^2 d\Omega \qquad (\sqrt{s} \gg m_e, m_\mu)$$

- Since we assume unpolarized beams and do not measure spin:

$$|\mathcal{M}|^2 \longrightarrow \underbrace{\frac{1}{2}\sum_r \frac{1}{2}\sum_{r'}}_{\text{average}} \underbrace{\sum_s \sum_{s'}}_{\text{sum}} |\mathcal{M}(r,r',s,s')|^2 = \dots$$

$$\dots = \frac{e^4}{4s^2} \underbrace{\sum_{s,s'} \left(\bar{u}_s(k)\gamma_\mu v_{s'}(k')\right)\left(\bar{u}_s(k)\gamma_\nu v_{s'}(k')\right)^*}_{\equiv A_{\mu\nu}} \underbrace{\left(\sum_{r,r'} \left(\bar{v}_{r'}(p')\gamma^\mu u_r(p)\right)\left(-\text{''}-\right)^*\right)}_{\equiv B^{\mu\nu}}$$

$$A_{\mu\nu} = \sum_{s,s'} \mathrm{tr}\left[\bar{u}_s(k)\gamma_\mu v_{s'}(k')\bar{v}_{s'}(k')\gamma_\nu u_s(k)\right] \quad \leftarrow \text{Thinking of spinors as } 1\times 4 \text{ \& } 4\times 1 \text{ matrices}$$

$$= \sum_{s,s'} \mathrm{tr}\left[u_s(k)\bar{u}_s(k)\gamma_\mu v_{s'}(k')\bar{v}_{s'}(k')\gamma_\nu\right]$$

$$= \mathrm{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right] = \mathrm{tr}\left[\not{k}\gamma_\mu \not{k}'\gamma_\nu\right] - m_\mu^2\,\mathrm{tr}\left[\gamma_\mu\gamma_\nu\right]$$

$$= 4\left(k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k') - m_\mu^2 \eta_{\mu\nu}\right) \qquad (\text{use Probl. 9.3})$$

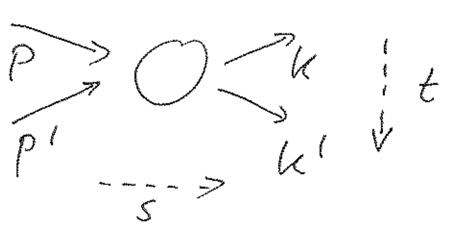
$$\boxed{\underline{\text{Recall:}}\ \mathrm{tr}\left[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right] = 4\left(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\sigma}\right)}$$

- Neglecting the m_μ^2-term since $m_\mu \ll \sqrt{s}$ (and of course also the corresponding m_e^2-term in $B_{\mu\nu}$) we find:

$$\frac{1}{4}\left(\sum\right)^4 |\mathcal{M}|^2 = \frac{e^4}{4s^2} \cdot 16\left(k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(kk')\right)\left(\cdots\{k,k' \to p,p'\}\cdots\right)$$

$$= \frac{8e^4}{s}\left((kp)(k'p') + (kp')(k'p)\right)$$

Reminder of <u>Mandelstam variables</u>:



$$s = (p+p')^2$$
$$t = (p-k)^2$$
$$u = (p-k')^2$$

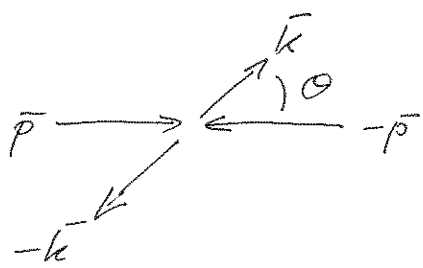
$$\left[s+t+u = \sum_{i=1}^{4} m_i^2 \right]$$

massless case: $p^2 = p'^2 = k^2 = k'^2 = 0 \Rightarrow s = 2pp' = 2kk'$

$$t = -2kp = -2k'p'$$
$$u = -2pk' = -2p'k$$

$$\Rightarrow \frac{1}{4}\left(\Sigma\right)^4 |\mathcal{M}|^2 = 2e^4 \, \frac{t^2+u^2}{s^2}$$

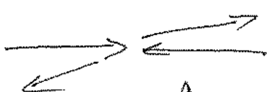
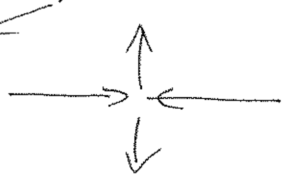
- Let us express this through the scattering angle θ in the centre-of-mass system (cms):



$$t = -2kp = -2(k_0 p_0 - \bar{k}\bar{p})$$
$$= -2k_0 p_0 (1-\cos\theta)$$
$$= -2\left(\frac{\sqrt{s}}{2}\right)^2 (1-\cos\theta) = -\frac{s}{2}(1-\cos\theta)$$

$$u = -s-t = -\frac{s}{2}(1+\cos\theta) \quad \Rightarrow \quad \frac{1}{4}(\Sigma)^4 |\mathcal{M}|^2 = e^4 (1+\cos^2\theta)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \cdot \frac{1}{4}(\Sigma)^4 |\mathcal{M}|^2 = \frac{\alpha^2}{4s}(1+\cos^2\theta) \quad ; \quad \alpha = \frac{e^2}{4\pi}$$

$\Rightarrow$ The process  is preferred by a factor of two w.r.t. . This can be understood in more detail (and more physically) as follows:

- Write $\sum |\mathcal{M}|^2$ as

$$\underbrace{\sum_{\text{spins}} (\bar{v}\gamma^{\mu}u)(\bar{v}\gamma^{\nu}u)^*}_{\sim B^{\mu\nu}} \eta^{\mu\mu'} \eta^{\nu\nu'} \underbrace{\sum_{\text{spins}} (\bar{u}\gamma^{\mu'}v)(\bar{u}\gamma^{\nu'}v)}_{\sim A^{\mu\nu}}$$

Can be replaced according to

$$\eta^{\mu\mu'} \to \eta^{\mu\mu'} - \frac{q^{\mu}q^{\mu'}}{q^2} \to \delta^{ii'}$$

since $q = p + p'$ and $\not{p}u(p) = 0$ etc.

- Thus (in the rest frame of the photon), we can restrict our "density matrices" A & B to their spatial components:

$$\rho_{ini}^{ij} \sim B^{ij} \; ; \; \rho_{fin}^{ij} \sim A^{ij}$$

$$\sum |\mathcal{M}|^2 \sim \rho_{ini}^{ij} \delta^{ii'} \delta^{jj'} \rho_{fin}^{i'j'}.$$

(The indices i, j correspond to the three physical polarizations of the intermediate, massive photon.)

- From $B^{\mu\nu} \sim p^{\mu}p'^{\nu} + p'^{\mu}p^{\nu} - \eta^{\mu\nu}(p \cdot p')$ it follows that $\rho_{ini}^{ij} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{ij} \sim \delta^{ij} - \hat{p}^i\hat{p}^j$

(in frame where $\bar{p} \parallel \hat{e}_3$, using $\hat{p} \equiv \bar{p}/|\bar{p}|$).

- Analogously, $\rho_{fin}^{ij} \sim \delta^{ij} - \hat{k}^i\hat{k}^j$

and $\sum |\mathcal{M}|^2 \sim \mathrm{tr}(\rho_{ini}\rho_{fin}) \sim 3 - 1 - 1 + (\hat{k}\cdot\hat{p})^2 = 1 + \cos^2\theta$

- The crucial physical point is that, as we have explicitly seen, the two incoming spin-1/2 particles always produce a spin-1 photon with spin ± 1 (<u>not 0</u>) along the beam axis. This is also true for the coupling of the decay products to the photon. Hence the correlation between beam & decay axis.

- So why this preference? After all $+\frac{1}{2} + (-\frac{1}{2}) = 0$ would also be a perfectly good process for producing a massive intermediate photon.

- The reason is that we have (due to the high-energy assumption) neglected the fermion mass. In this limit, we are dealing with two indep. types of fields

$$\bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R ,$$

each with conserved $U(1)$ charge.

- e_L^+, e_L^- can annihilate: $p \underset{\vec{s}}{\longrightarrow} \ \underset{\vec{s}}{\longleftarrow} p'$

- e_R^+, e_R^- can annihilate: $p \underset{\overleftarrow{s}}{\longrightarrow} \ \underset{\overleftarrow{s}}{\longleftarrow} p'$

- e_R^+, e_L^- can not! $\Rightarrow$ no spin-0 (along 3-axis) photon can be produced.

<u>"No helicity flip in absence of mass term!"</u>