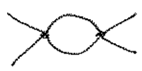
12 Renormalization

12.1 The Concept

- Let us call $\{Q_i\}$ $(i = 1, 2, \cdots)$ the set of quantities we would like to calculate in a given QFT.

  [$\{Q_i\}$ = { cross-sections, Greens fcts., self-energies, ... } ;
  $\supset$ { observables }]

- We already know that, at higher orders in pert. theory, divergent loop integrals appear.

  [e.g. $2 \to 2$ scatt. in $\lambda\varphi^4$-theory ;  +  + ⋯ ;

   $\sim \int d^4k \, \frac{1}{k^2 - m^2 + i\varepsilon} \cdot \frac{1}{(k+q)^2 - m^2 + i\varepsilon}$]

- Let us, for the moment, regularize by analytical cont. to euclidean momenta ($k \to k_E$; $k_E^2 = k_0^2 + k_1^2 + \cdots$) and introducing a cutoff: $|k_E| \leq \Lambda$.

- As a result, $Q_i = Q_i(\Lambda)$ and the limit $\Lambda \to \infty$ can naively not be taken. Renormalization is the method of properly taking this limit nevertheless.

- Our example here will be QED. We write our familiar Lagrangian as

$$\mathcal{L} = -\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i(\partial\!\!\!/ + i e_0 A\!\!\!/_0) - m_0)\psi_0 \quad ; \quad F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$$

and now call our familiar fields & couplings "bare" (hence "0").

- We can rewrite them in terms of "renormalized" quantities:

$$A_0^\mu = Z_A^{1/2} A^\mu \;;\; \psi_0 = Z_\psi^{1/2} \psi \;;\; e_0 = Z_e e \;;\; m_0 = Z_m m$$

such that

$$\mathcal{L} = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + Z_\psi \bar{\psi} (i(\partial\!\!\!/ + i Z_e Z_A^{1/2} e A\!\!\!/)) - Z_m m)\psi$$

- <u>Crucial idea:</u> Choose Z_i to be specific fcts. $Z_i(\Lambda)$ of the cutoff such that $Q_i = Q_i(e, m, \Lambda, Z_i(\Lambda))$ have a well-defined (finite) limit as $\Lambda \to \infty$:

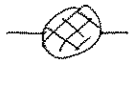
$$Q_i^\infty \equiv \lim_{\Lambda\to\infty} Q_i(e, m, \Lambda, Z_i(\Lambda)).$$

- If this is possible, the theory is called <u>renormalizable</u>. This property is highly non-trivial since there are only <u>finitely</u> many Z's, but <u>infinitely</u> many Q_i.
- Even if possible, the above procedure is clearly non-unique since one can always "move finite factors between e.g. Z_e and e. $\Rightarrow$ Need to impose "<u>renormalization conditions</u>". [E.g. impose certain values for poles & residues of all propagators and as many cross-sections as there are indep. couplings. Only <u>further</u> cross sections and correl. fcts. will then be <u>predictions</u> of <u>the theory</u>.]

<u>Comment:</u> If one accepts that corr. fcts. continue to be divergent in the limit $\Lambda \to \infty$ and insists only on a finite limit for observables, $\{O_i\} \subset \{Q_i\}$, fields do not need to be renormalized. Just $Z_e = Z_e(\Lambda)$ & $Z_m = Z_m(\Lambda)$ will do the job.

12.2 Renormalization conditions

① For completeness (though we don't need it in QED) we start with mass & field normalization of a scalar.

Recall:  $= -i\Pi(p^2)$;  $= \dfrac{i}{p^2 - m^2 - \Pi(p^2)}$

(Here m is the renormalized mass parameter, appearing in the renormalized Lagrangian, as in QED in 12.1.) We called that mass "m_0" in Sect. 7.6, but now we used up "m_0" for the bare mass.)

<u>Mass:</u> $m_{phys}^2 = m^2 + \Pi(m_{phys}^2)$, as derived in Sect. 7.6

Our proposed choice of condition is $\boxed{m^2 = m_{phys}^2}$

or, equivalently $\underline{\underline{\Pi(m^2) = 0.}}$

<u>Field (or "wave fct") normalization:</u>

$Z^{-1} = 1 - \Pi'(m_{phys}^2)$. Our choice: $\boxed{Z = 1}$ or, equivalently $\underline{\underline{\Pi'(m^2) = 0}}$.

Excursion into scalar case (to make the above more tangible):

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 Z_\varphi - \frac{1}{2}\varphi^2 m^2 Z_\varphi Z_m - \frac{\lambda}{4!}\varphi^4 Z_\lambda Z_\varphi^2$$

let $Z_\varphi = 1 + \delta Z_\varphi$ etc.

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 + \underbrace{\frac{1}{2}(\partial\varphi)^2\delta Z_\varphi - \frac{1}{2}m^2\varphi^2(\delta Z_\varphi + \delta Z_m)}_{\text{This is "}O(\lambda)\text{" and is hence treated as an interaction}} + \cdots$$

$\Rightarrow$ Feynman rule $-\times-$

$-\times- = -im^2(\delta Z_\varphi + \delta Z_m) + \cdots$

$\Rightarrow i\Pi(p^2) = -\times- + \underbrace{}_{\sim\Lambda^2}$  at $O(\lambda)$

We see that get conditions of the type $\delta Z \sim \lambda f(\Lambda)$, with $f(\Lambda) \to \infty$ as $\Lambda \to \infty$. The logic in pert. theory is to always take the limit $\lambda \to 0$ (by def. of pert. th.) more seriously than $\Lambda \to \infty$. In other words, δZ is a "small correction" in spite of Λ.
Only after all Λ-dependence has disappeared, are we allowed to give λ its "measured" physical value

(2) Mass- & field normaliz. for the electron:

a  b $= -i\Sigma(\not{p})_a{}^b$;  $= \dfrac{i}{\not{p} - m - \Sigma(\not{p})}$

[Here the self-energy Σ is defined in complete analogy to the scalar case. It is, of course, a matrix since the field has 4 components. Also, it is not a Lorentz-invariant. Hence, it can depend on p more generally than through p^2. This is accounted for by giving Σ the argument $\not{p}$.]

In analogy to the scalar case we have:

Mass: $\boxed{m = m_{phys}}$ $\longrightarrow$ $\Sigma(m) = 0$

(our condition) (realized by..)

Field: $\boxed{Z = 1}$ $\longrightarrow$ $\Sigma'(m) = 0$

(our condition) (realized by..)

Once $\Sigma(m) = \Sigma'(m) = 0$ holds, it is easy to see that the propagator does indeed have a pole at $p^2 = m^2$ with "canonical" residue

Taylor expand: $\Sigma(\not p) = \underset{\substack{\uparrow \\ 0}}{\Sigma(m)} + \underset{\substack{\uparrow \\ 0}}{\Sigma'(m)}(\not p - m) + \frac{1}{2}\Sigma''(m)(\not p - m)^2 + \dots$

$$\Rightarrow \frac{i}{\not p - m - \Sigma(\not p)} = \frac{i}{(\not p - m)\left(1 - \frac{1}{2}\Sigma''(m)(\not p - m) + \cdots\right)}$$

$$= \frac{i(\not p + m)}{(p^2 - m^2)\left(1 - \frac{1}{2}\Sigma''(m)(\not p - m) + \cdots\right)}$$

$$= \frac{i(\not p + m)\left(1 + \frac{1}{2}\Sigma''(m)(\not p - m) + \cdots\right)}{p^2 - m^2}$$

$$= \underbrace{\frac{i(\not p + m)}{p^2 - m^2}}_{\substack{\text{pole at } m^2;\ \text{residue same} \\ \text{as in free case.}}} + \underbrace{i\left[\frac{1}{2}\Sigma''(m) + \cdots\right]}_{\substack{\text{analytic at } p^2 = m^2;\ \text{hence} \\ \text{no contribution to residue}}}$$

③ Field normalization for <u>photon</u>

(the mass should stay zero automatically, by the structure of the theory)

μ  ν $= i\Pi_{\mu\nu}(q)$

It is (like in the electron case) useful to think of the propagator and hence of $\Pi_{\mu\nu}$ as of a 4x4 matrix. Lets call this matrix Π^M:

 $= i\Pi^M$;  $= \dfrac{i}{-q^2\eta + \Pi^M(q)}$

$$\left[\frac{i}{-q^2\eta} + \frac{i}{-q^2\eta}\, i\Pi^M \frac{i}{-q^2\eta} + \cdots\right]$$

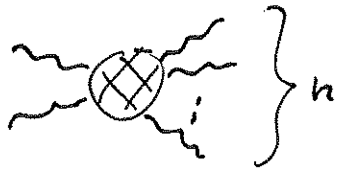
matrix $\eta_{\mu\nu}$

- We know that $\Pi^{\mu}(0) = 0$ must be maintained for the photon to stay massless "at higher order". We will assume that our regularization will always respect this.
- By covariance, $\Pi_{\mu\nu}(q) = \eta_{\mu\nu} A(q^2) + q_\mu q_\nu B(q^2)$, in full generality.
- Gauge invariance enforces $\underline{\underline{A = B \cdot q^2}}$, i.e.

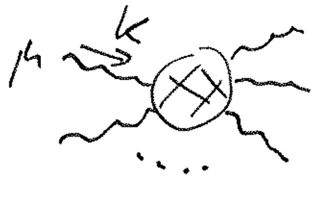
$$\Pi_{\mu\nu}(q^2) = (\eta_{\mu\nu} q^2 - q_\mu q_\nu)\,\Pi(q^2)$$

We will give a general & careful argument for this in a moment. A quick argument is as follows:

- Consider $2 \to n$ photon scattering, based on the amplitude



- Let's change one of the polariz. vectors of the ext. photons according to $\varepsilon^\mu(k) \to \varepsilon^\mu(k) + \alpha k^\mu$.
- The amplitude should not change since this is just a change of gauge, i.e.



$$\cdot\, k^\mu = 0.$$

- In the special case of just two ext. lines this says

$\cdot\, k^\nu = 0$ or $\Pi_{\mu\nu}(q) q^\nu = 0$

$\Rightarrow A = B \cdot q^2$

- Thus, we have

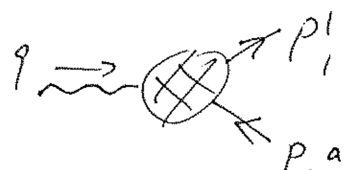
$$= \frac{i}{-\eta q^2(1-\Pi(q^2)) - \underbrace{(q\otimes q)\cdot\Pi(q^2)}_{\substack{\text{does not contribute if}\\ \text{contracted with phys. polariz.}\\ (\varepsilon^\mu(q)\cdot q_\mu = 0)}}}$$

$\Rightarrow$ Our condition: $\boxed{Z=1}$ $\longrightarrow$ realized by: $\underline{\Pi(0)=0}$.

(where  $\equiv i\Pi_{\mu\nu}(q) \equiv (\eta_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2)$; note that due to the extracted prefactor q^2, our $\Pi(0)=0$ here is analogous to the "$\Pi'(m^2)=0$" in the scalar case)

④ <u>Vertex normalization</u>

(This is technically simpler but in spirit analogous to fixing some specific cross section.)

 $\equiv ie\Gamma^\mu(p,p')_b{}^a$, with p,p' on-shell

Our condition: | <u>At $q\to 0$</u>, this should be the <u>same as at tree-level</u>, i.e.

$$\underline{\Gamma^\mu(p,p) = \gamma^\mu}$$

It is interesting to see, how many conditions we actually impose – this is non-trivial since Γ^μ is in general a matrix!

- In full generality: $\Gamma^\mu = \gamma^\mu\cdot A(p,p') + (p'^\mu + p^\mu)\cdot B(p,p') + \ldots$

$\ldots + (p'^\mu - p^\mu) \cdot C(\not p, \not p')$

(since p, p' are the only indep. vectorial arguments.)

- Since Γ^μ always appears between $\bar{u}(p')$ & $u(p)$ (or $\bar{v}, v$), and $\bar{u}(p')\not p' = \bar{u}(p') \cdot m$, $\not p u(p) = m u(p)$, we can w.l.o.g. assume A, B, C to be numbers $\times \mathbb{1}$:

$$\Gamma^\mu = \gamma^\mu A(q^2) + (p'^\mu + p^\mu) B(q^2) + (p'^\mu - p^\mu) C(q^2)$$

this is the only kinemat. invariant

- As before, by gauge independence,

$\boxed{\bar{u}(p') q_\mu \Gamma^\mu u(p) = 0}$ (even if $q^2 \neq 0$ since the gauge parameter could be in the propagator: )

$q_\mu \cdot (p'^\mu - p^\mu) = q^2 \neq 0$ in general $\Rightarrow C = 0$

$q_\mu (p'^\mu + p^\mu) = p'^2 - p^2 = 0 \Rightarrow B$ unconstrained.

$$\Gamma^\mu = \gamma^\mu A(q^2) + (p'^\mu + p^\mu) \cdot B(q^2)$$

$\boxed{\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]}$

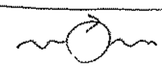
<u>equivalent:</u> $\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m^2} F_2(q^2)$

(pure γ-algebra)

"form factors"

Our condition was just $\underline{F_1(0) = 1}$ (only <u>one</u> constraint)

12.3 QED β-function

- In principle, we could now put all of this together to calculate two cross-sections, one being used to fix e, and interpret the second as a one-loop prediction of QED (For the first, one typically chooses, $e\gamma \to e\gamma$ at $k \to 0$. With very little extra work, one finds that this simply identifies our "e" with the class. charge of the electron.)
- However, we would need at least , , , + whichever diagrams are needed for our "proper" cross-section. We do not have time for this.
- However, there is one very important and physical quantity which we can obtain based on just one diagram: The β-fct., which determines the energy-dependence of the coupling "constant" e_0.
- Indeed, by def. our renormalized coupling is indep. of Λ. Since $e_0(\Lambda) = Z_e(\Lambda) \cdot e$, we have

$$\frac{d}{d\ln\Lambda} e_0(\Lambda) = e \frac{d}{d\ln\Lambda} Z_e(\Lambda) = e \frac{d}{d\ln\Lambda}\left(1 + \delta Z_e\right)$$

$$\underset{\substack{\uparrow \\ \text{LO approx.}}}{\simeq} e \frac{d}{d\ln\Lambda}\left(1 + \underset{\substack{\uparrow \\ \text{coeff. of 1-loop divergence} \\ \text{in } \delta Z_e \text{ (just a number} \\ \text{to be determined)}}}{c} \cdot e^2 \ln\Lambda\right) = c \cdot e^3$$

- Since $e \simeq e_0$ at LO, we can disregard higher-order terms on the r.h. side and write

$$\frac{d}{d\ln\Lambda} e_0(\Lambda) \simeq c \cdot e_0(\Lambda)^3.$$

- By def., $\boxed{\beta(e_0(\Lambda)) \equiv \frac{d}{d\ln\Lambda} e_0(\Lambda)}$, so the above gives us the LO β-fct. for the bare coupling of QED.

[Note that, given $\beta(e_0)$, the above <u>diff. equation</u> (an "RGE" or "Renormalization group equation") determines e_0 for <u>any</u> Λ, given some initial condition.]

- Why is this "running" of the bare coupling of any relevance?
- Let us define some "scale-dep. phys. coupling" as follows: Calculate a cross-sect. at given energy $\sqrt{s}$ at LO,

$$\frac{d\sigma}{d\Omega} = \frac{c_1 \cdot e_0^4(\Lambda)}{s},$$

where we used the bare coupling, which is OK since the high-order difference is small as long as $\ln(\Lambda^2/s)$ is not large. (cf. $\delta Z \sim e^2 \ln(\Lambda/\sqrt{s})$).

- Let us <u>define</u> a "scale-dep. phys. coupling" by

$$e^4(\mu) \equiv \frac{s}{c_1} \cdot \frac{d\sigma}{d\Omega}\Big|_{s=\mu^2}.$$

- We see that $e(\mu) \simeq e_0(\Lambda)$ at $\mu \simeq \Lambda$.

Hence:
$$\boxed{\frac{d}{d\ln\mu}\, e(\mu) = \beta(e(\mu)) \;;\quad \beta(e) = c\, e^3}$$
$$\boxed{\text{with } \delta Z_e = c\, e^2 \ln\Lambda}$$

Now let's calculate:

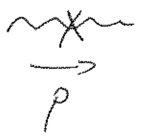
- First note that the structure $\partial_\mu + ieA_\mu$ will be unchanged under renormalization if $\underline{Z_e Z_A^{1/2} = 1}$. This is indeed exactly true as can be shown using "Ward-Takahashi-identities" (see below).

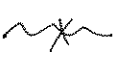
> Note: Many books use the notation
> $Z_A \equiv Z_3 \;;\; Z_\psi = Z_2 \;;\; Z_e Z_\psi Z_A^{1/2} = Z_1$.
> Then $Z_e Z_A^{1/2} = 1 \longrightarrow \underline{\underline{Z_1 = Z_2}}$

- We now know that $c = -\frac{1}{2}\cdot\frac{1}{e^2}\frac{d}{d\ln\Lambda} Z_A$, so all we need is the $\ln\Lambda$-term in δZ_A.

- δZ_A corresponds to $\mathcal{L} \supset -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,\delta Z_A$
$$= \frac{1}{2}\left(A_\mu \partial^2 A^\mu - A_\mu \partial^\mu \partial^\nu A_\nu\right)\delta Z_A$$

- This gives a "vertex":

 $= i\left(-\eta^{\mu\nu} p^2 + p^\mu p^\nu\right)\cdot \delta Z_A$

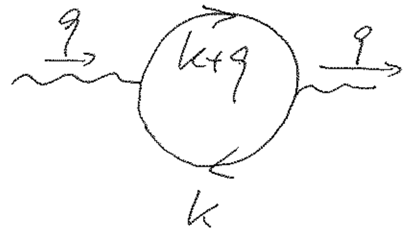
- Hence $\underbrace{i\Pi_{\mu\nu}}$ =  + $\underbrace{\text{}}$ 

"one-loop"

$\equiv i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)$ $\qquad\qquad \equiv i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi_{(1)}(q^2)$

Thus: $c = -\frac{1}{2e^2}\cdot\left\{\text{coeff. of } \ln\Lambda\text{-term in } \Pi_{(1)}(q^2)\right\}$

12.4 Vacuum polarization in dimensional regularization

$i\Pi^{\mu\nu}_{(1)}(q) =$  $= (-1)(ie)^2\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\left[\gamma^\mu\frac{i}{\not k-m}\gamma^\nu\frac{i}{\not k+\not q-m}\right]$

$0,1,\dots,d-1$

because our whole theory is now def. in d space-time dimensions

Jumping slightly ahead, let us give away the basic underlying idea:

$$\int^\Lambda\frac{d^4k_E}{(k_E^2+m^2)^2} = \Omega_3\int_0^\Lambda\frac{d|k_E|\cdot|k_E|^3}{(|k_E|^2+m^2)^2} \simeq \Omega_3\cdot\ln(\Lambda/m)$$

$\Big\downarrow$ by dim. reg., $d = 4-\epsilon$

$$\dots\simeq\int_m^\infty\Omega_3\frac{d|k_E|\cdot|k_E|^{3-\epsilon}}{|k_E|^4} \simeq \Omega_3 m^{-\epsilon}\int_1^\infty\frac{dx}{x^{1+\epsilon}} = \Omega_3 m^{-\epsilon}\frac{1}{\epsilon}$$

$\Rightarrow$ The poles in $\epsilon = 4-d$ track the physical log-divergences. This can be made mathematically fully rigorous by thinking of d as a complex variable, which makes all our expressions well-defined except at certain isolated points.

Crucial: Poinc./gauge inv. fully preserved by dim. reg.

- $\Pi_{\mu\nu} = (q^2\eta_{\mu\nu} - q_\mu q_\nu)\cdot \Pi$

$$\Pi_\mu{}^\mu = (d\cdot q^2 - q^2)\cdot \Pi \quad \Rightarrow \quad \Pi = \frac{1}{(d-1)\cdot q^2}\,\Pi_\mu{}^\mu .$$

- $$\mathrm{tr}\left[\gamma^\mu \frac{i}{\not k - m}\gamma_\mu \frac{i}{\not k + \not q - m}\right] = -\frac{\mathrm{tr}\left[\gamma^\mu(\not k + m)\gamma_\mu(\not k + \not q + m)\right]}{(k^2-m^2)((k+q)^2-m^2)}$$

$$= -\frac{\mathrm{tr}\left[((2-d)\not k + m\cdot d)(\not k + \not q + m)\right]}{(k^2-m^2)((k+q)^2-m^2)} = 4\cdot\frac{(d-2)k\cdot(k+q) - m^2 d}{(k^2-m^2)((k+q)^2-m^2)}$$

Using Cliff.-alg. in d dims.; where $\gamma^\mu\gamma_\mu = d$;
$\gamma^\mu \not k \gamma_\mu = 2\not k - \gamma^\mu\gamma_\mu \not k = (2-d)\not k$, etc.

Note: We also used $\mathrm{tr}(\mathbb{1}) = 4$, which is not obviously right, but in our case does not matter...

Note: $\int d^dk$ of the above let's one expect a quadratic divergence at $d=4$. In dim. reg. this corresponds to a pole at $d=2$. But we see that at $d=2$ the coeff. of the k^2-term vanishes. Hence, $\Pi_\mu{}^\mu$ is not quadr. divergent. That's not so easy to see with a cutoff Λ.

- Next, use the so-called "Feynman parameter" x:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \qquad \text{(Check this!)}$$

$$\Rightarrow i\Gamma_{(1)\mu}{}^{\mu} = 4e^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \, \frac{(d-2)\, k\cdot(k+q) - m^2 d}{\left[(1-x)(k^2-m^2) + x\left((k+q)^2-m^2\right)\right]^2}$$

Now change order of integration and substitute integration variable: $k = k' - xq$ & rename $k' \equiv k$.

$\Rightarrow$ denominator: $\Big[k^2 + \underbrace{x(1-x)q^2 - m^2}_{\equiv -\Delta \text{ (convenient abbreviation)}}\Big]^2$

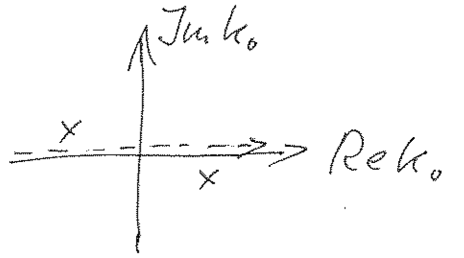
$\Rightarrow$ numerator: $(d-2)\left\{k^2 + \underbrace{(1-2x)k\cdot q}_{} - x(1-x)q^2\right\} - m^2 d$

This term is odd w.r.t. $k \to -k$ & hence vanishes under integration

$\Rightarrow$ Together:

$$i\Gamma_{(1)\mu}{}^{\mu} = 4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \cdot \frac{(d-2)(k^2 - x(1-x)q^2) - m^2 d}{(k^2-\Delta)^2}$$

- We have all the time for brevity suppressed the "$i\varepsilon$". But in fact it is still there ($m^2 \to m^2 - i\varepsilon$), such that the pole structure in the k_0-plane is



$\Rightarrow$ We can rotate our integr. contour from real to imag. axis w/o encountering poles according to $k_0 \to ik_0$.

$\Rightarrow dk_0 \to i\,dk_0 \;;\; k^2 = k_0^2 - \vec{k}^2 \longrightarrow -k_E^2 = -k_0^2 - \vec{k}^2$

↑ Euclidean momentum

$$i\Gamma_{(1)\mu}{}^{\mu} = 4ie^2 \int \frac{d^dk_E}{(2\pi)^d} \int_0^1 dx \, \frac{(d-2)(-k_E^2 + x(1-x)q^2) - m^2 d}{(k_E^2+\Delta)^2}$$

- The term $\frac{k_E^2}{(k_E^2+\Delta)^2}$ can be rewritten as

$$\frac{k_E^2+\Delta-\Delta}{(k_E^2+\Delta)^2} = \frac{1}{(k_E^2+\Delta)^1} - \frac{\Delta}{(k_E^2+\Delta)^2}.$$

- Hence we only need the single integral (with $n=1,2$ in our case):

$$\int \frac{d^d k_E}{(2\pi)^d} \cdot \frac{1}{(k_E^2+\Delta)^n} = \underbrace{\int \frac{d\Omega_{d-1}}{(2\pi)^d}}_{\substack{\text{well-def. for} \\ \text{all integer } d>1}} \cdot \underbrace{\int_0^\infty dk_E \frac{k_E^{d-1}}{(k_E^2+\Delta)^n}}_{\text{well-def. for all } d<2n}$$

$\Downarrow$ easily promoted to analyt. fct. of d (with poles) using

$$\int d\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$\Downarrow$ easily promoted to analytic fct. of d & Δ (with poles)

$$\ldots = \frac{\Gamma(d/2)\Gamma(n-d/2)}{2\Gamma(n)} \cdot \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

- Crucially, for $n=2$ & $d=4-\epsilon$ we get a pole:

$$\Gamma(n-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$$

($\gamma = 0.577\ldots$, Euler-const.)

- Crucially, we do not get a pole at $d=2$ (corr. to a quadratic divergence) from the k_E^2-term ($\equiv$ the $(n=1)$ term):

$$\ldots (d-2) \cdot \frac{\Gamma(1-d/2)}{\Gamma(1)} \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} = -2 \frac{(1-\frac{d}{2})\Gamma(1-\frac{d}{2})}{1} \cdot \Delta^{\epsilon/2}$$

$$= -2\Gamma(2-\tfrac{d}{2})\Delta^{\epsilon/2} \implies \text{also pole at } d=4.$$

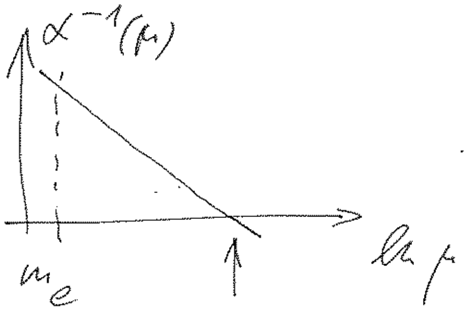
- put everything together
- perform x-integration (easy)

$$\Rightarrow \Pi_{(1)}(q^2) = \frac{1}{(d-1)q^2} \Pi_{(1)\mu}{}^{\mu} \Bigg|_{\substack{\text{at } q^2=0 \\ \& \ \epsilon \to 0}} = -\frac{e^2}{6\pi^2\epsilon}$$

$$\Rightarrow c = \frac{1}{12\pi^2} \quad ; \quad \underline{\underline{\beta(e) = \frac{e^3}{12\pi^2}}} \quad \text{at 1-loop}$$

$$\frac{d}{d\ln\mu}\Big(\underbrace{\frac{1}{e^2}}_{\text{"}\sim\alpha^{-1}\text{"}}\Big) = -\frac{1}{e^3}\beta(e) = -\frac{1}{12\pi^2}$$

$\Rightarrow$

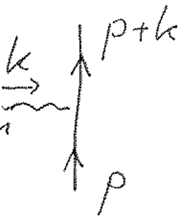


"Landau pole" (but only at very large energy)

12.5 The Ward-Takahashi identity

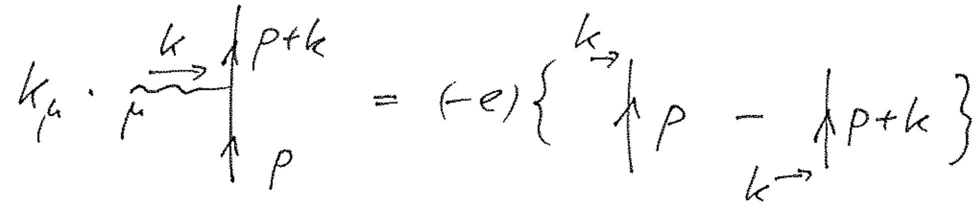
... is an identity between amplitudes or Green's-fcts. which relies on gauge invariance.

- We start with an illustrative example calculation.
- Consider LO 3-point fct. with γ-propagator amputated but fermion propagator present. Contract with k_μ:

$k^\mu \cdot$ 

$$= \frac{i}{(\not{p}+\not{k})-m}\, ie\underbrace{\not{k}}_{=((\not{p}+\not{k})-m)-(\not{p}-m)} \frac{i}{\not{p}-m} = (-e)\left\{\frac{i}{\not{p}-m} - \frac{i}{(\not{p}+\not{k})-m}\right\}$$

<u>Pictorially:</u>



<u>In words:</u> An ext. γ-line contracted with k_μ can be removed & the vertex replaced by the diff. of the two propagators before & after the vertex (multipl. by $-e$).

- We can also amputate the ext. ferm. props. by multiplying with $-((\not p+\not k)-m)\{\cdots\}(\not p-m)$:

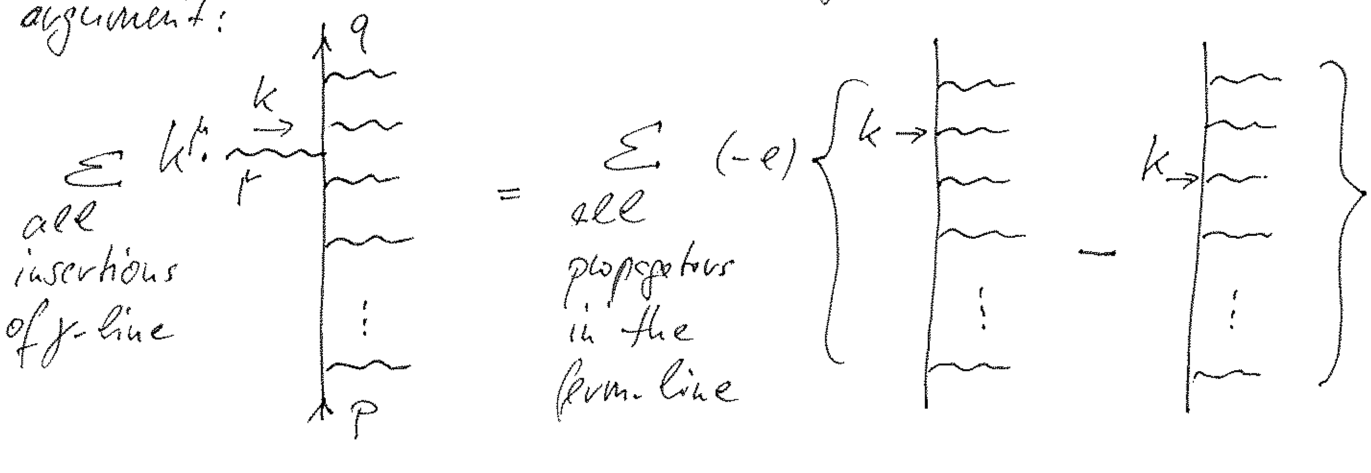
$$\Rightarrow \quad -e k_\mu \Gamma^\mu = i\{((\not p+\not k)-m)-(\not p-m)\}$$

in our LO case just γ^μ

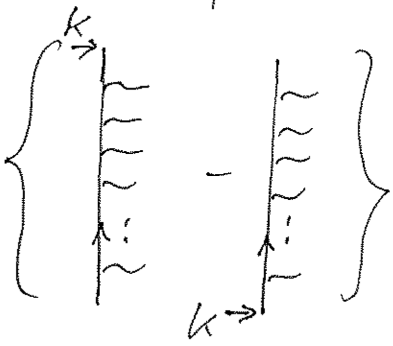
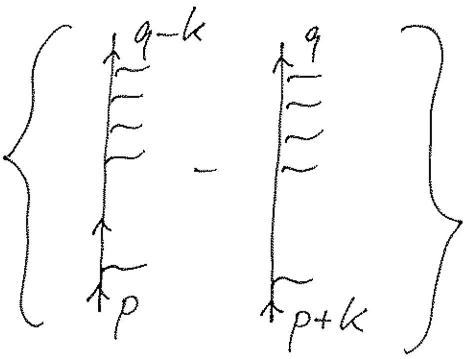
- Further, we can look at the actual matrix element by putting p & $p+k$ on-shell and multiplying with $\bar u(p+k)\cdots u(p)$: $\Rightarrow \bar u(p+k)k_\mu\Gamma^\mu u(p)=0$.

(This is of course just a special case of the general gauge-inv. based claim $k_\mu \mathcal{M}^\mu=0$ for any <u>phys.</u> amplitude with ext. photon.)

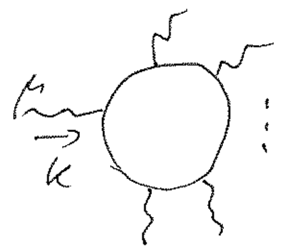
- The key interest is in the simple generalization of our argument:



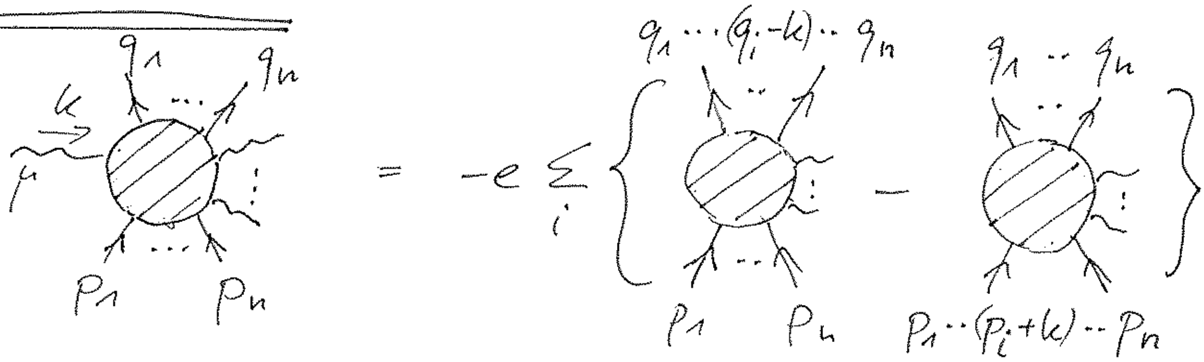
- In the complete sum, all terms except the last & first cancel pairwise. Thus:

$\dots = (-e)$  $= (-e)$ 

- The same argument can be made for a closed loop, in which case first & last term also cancel after an appropr. shift of the integration variable.

$\sum_{\substack{\text{all}\\ \text{insertions}}} k_\mu$  $= 0$

$\Rightarrow$ General theorem:

k_μ 

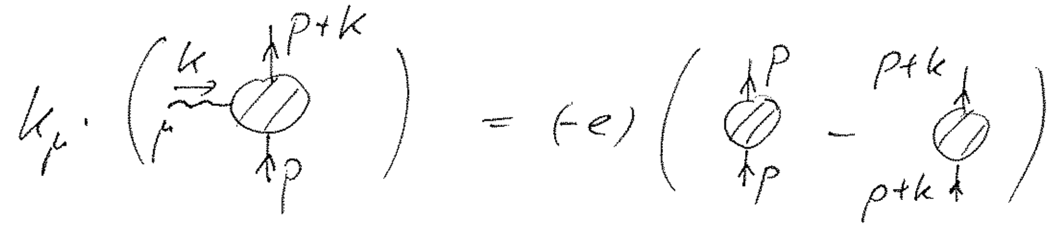
[Only γ-line amputated!]

① Corollary: Amputate the ferm.-props. ($\times \not{q}_i$ & $\times \not{p}_i$), go on-shell, multiply with ext. spinors ($\bar{u}(q_i)$; $u(p_i)$).

$\Rightarrow$ Find zero on r.h. side since e.g. $\bar{u}(q_i)\not{q}_i \frac{i}{\not{q}_i - \not{k}} = 0$.

$$\Rightarrow \quad k_\mu \mathcal{M}^\mu = 0$$

② Corollary: (Simplest, 3-point version of the general case.)
often just this is called W.-T.-identity



or, with $S(p) = \frac{i}{\not p - m - \Sigma(\not p)}$,

$$S(p+k)\left(iek_\mu \Gamma^\mu(p+k,p)\right)S(p) = (-e)\left(S(p) - S(p+k)\right)$$

$$-ik_\mu \Gamma^\mu(p+k,p) = S^{-1}(p+k) - S^{-1}(p)$$

$\Rightarrow$ Divergence in $k_\mu \Gamma^\mu$ at $k \to 0$ = Divergence in Σ'

$\Rightarrow$ We can (and must, if we want to keep explicit gauge invariance) choose $Z_1 = Z_2$.

$$\left(Z_\psi Z_A^{1/2} Z_e = Z_\psi \right)$$

12.6 Sketch of an operator derivation of the W-T-identities

- Recall that $j_\mu(x) = \bar\psi(x)\gamma_\mu \psi(x)$; $\partial^\mu j_\mu = 0$
- Consider $\partial^x_\mu \langle 0| T j^\mu(x)\psi(y)\bar\psi(z)A_\nu(w)|0\rangle$

$$= \langle 0| T\left\{ [j^0(x),\psi(y)]\,\delta(x^0-y^0)\,\bar\psi(z) + \psi(y)\,[j^0(x),\bar\psi(z)]\,\delta(x^0-z^0)\right\}A_\nu(w)|0\rangle$$

$\uparrow$ Here we re-wrote the T-ordering using Θ-fcts. ; evaluated ∂^x_0 using the product rule ; used $\partial_t \Theta(t) = \delta(t)$ and $\partial^\mu j_\mu = 0$.

- Next, we recall that $Q = \int d^3x\, j^0(x)$ generates the symm. which underlies the existens of the cons. current j. In our case

$$\psi \to e^{-i\alpha}\psi \;;\; \delta_\alpha \psi = -i\alpha\psi \;;\; \delta_\alpha \xrightarrow[\text{QM}]{} \underbrace{i\alpha Q}_{\text{Symm. trf. acting on Hilbert space}}$$

• The corresponding action on field operators is

$$\psi \to e^{i\alpha Q}\psi e^{-i\alpha Q} \quad \text{or} \quad \delta_\alpha \psi = [i\alpha Q, \psi] \stackrel{!}{=} -i\alpha\psi$$

$\Rightarrow [Q, \psi] = \psi$ or, field-theoretic / current version:

Follows explicitly from commut. relations

$$\left\{ \begin{aligned} \delta(x^0-y^0)[j_0(x), \psi(y)] &= -\psi(x)\,\delta^4(x-y) \\ \delta(x^0-y^0)[j_0(x), \bar\psi(y)] &= \bar\psi(x)\,\delta^4(x-y) \\ \delta(x^0-y^0)[j_0(x), A_\nu(y)] &= 0. \end{aligned} \right.$$

• Applying this, we find:

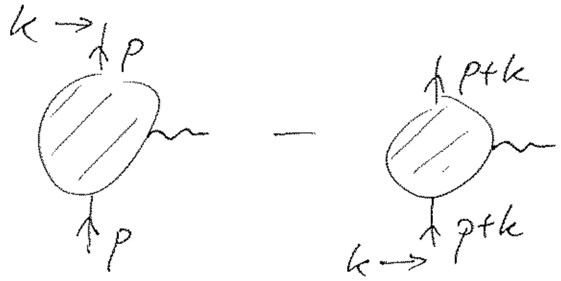
$$\partial^x_\mu \langle T j^\mu(x)\psi(y)\bar\psi(z)A_\nu(w)\rangle$$

$$= \langle T\psi(y)\bar\psi(z)A_\nu(w)\rangle \cdot \{-\delta^4(x-y) + \delta^4(x-z)\}$$

$\bar\psi\gamma^\mu\psi$

... $\Downarrow$ ... after Fourier-trfs. ..... $\Downarrow$ ..

$k^\mu \cdot$ 



• This easily generalizes to many fermion & photon lines and provides a formal proof of the W-T-identities.

(cf. Itzykson / Zuber, Chapter 8.4.1)