13. Non-abelian gauge theories and the Standard Model

13.1 Non-abelian gauge theories

- Recall how we derived abelian gauge theories from demanding invariance of $\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi \phi^*$ under $\phi(x) \to e^{i\alpha(x)} \phi(x)$.
- Now, let $\phi(x) \in V$ (vector space) such that $\mathcal{L}$ is inv. under group G with repr. R acting on V:

$$\phi(x) \to R(g)\,\phi(x) \quad ; \quad \text{then let } g \to g(x)$$

("gauge" inv.).

- To be concrete, focus on $G = SU(N)$; $R = F$ ("fundamental")
- Then: $\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi^i - m^2 \phi_i^* \phi^i = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi$.
- This is obviously inv. under

$$\phi \to U\phi \quad (U \in SU(N)), \text{ i.e. } \phi^i \to U^i{}_j \phi^j$$

because $U^\dagger U = \mathbb{1}$.

- Let us, as in the abelian case, introduce

$$D_\mu \phi = \partial_\mu \phi + i \underset{\substack{\uparrow \\ \text{matrix!}}}{A_\mu} \phi \quad \left(\text{or } (D_\mu \phi)^i = \partial_\mu \phi^i + i A_\mu{}^i{}_j \phi^j\right),$$

and demand $D_\mu \phi \to U D_\mu \phi$ <u>even</u> if $U = U(x)$.

(This will obviously ensure invar. of $\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \ldots$)

- If we can ensure $D'_\mu = U D_\mu U^+$, we are done.

(Since $(D_\mu \phi)' = D'_\mu \phi' = U D_\mu U^+ U \phi = U(D_\mu \phi))$.

- Thus, we need

$$\mathbb{1}\cdot\partial_\mu + iA'_\mu \stackrel{!}{=} U(\mathbb{1}\cdot\partial_\mu + iA_\mu)U^+$$

We will always suppress this "$\mathbb{1}$" below.

- Crucially, the above has to be read as an equality of diff. operators. Hence:

$$\partial_\mu + iA'_\mu = \partial_\mu + U(\partial_\mu U^+) + UiA_\mu U^+$$

$$\text{or:} \quad \underline{\underline{A'_\mu = UA_\mu U^+ - iU(\partial_\mu U^+)}}.$$

- The infinites. version is obtained by assuming $U = e^{iT}$ & $T \in \text{Lie}(G)$ an infinitesimal Lie-alg. element. [More correctly: $U = e^{i\epsilon T}$, $iT \in \text{Lie}(G)$, ϵ infinitesimal.]
- With $U = \mathbb{1} + iT + \cdots$ we get:

$$\underline{\underline{\delta A_\mu = i[T, A_\mu] - \partial_\mu T}} \quad (+\,O(T^2))$$

- We see that if $A_\mu \in \text{Lie}(G)$ to start with, it will remain so under gauge trfs. This completes our construction:

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \qquad (\text{for } SU(N)\ \&\ R = F)$$

$$= -\frac{1}{2g^2}\,\text{tr}(F_{\mu\nu}F^{\mu\nu}) + (D_\mu\phi)^+(D^\mu\phi) - m^2\phi^+\phi$$

$$F_{\mu\nu} = -i\,[D_\mu, D_\nu]\ ;\quad D_\mu = \partial_\mu + iA_\mu\ ;\quad A_\mu \in \text{Lie}(G).$$

- Invariance straightforwardly follows from $D'_\mu = U D_\mu U^+$. Indeed, it is easy to check that $F_{\mu\nu}$, defined as a diff. operator, happens to be just a matrix. ($\in \mathrm{lie}(G)$)
  Since, obviously, $F'_{\mu\nu} = U F_{\mu\nu} U^+$, $\mathrm{tr}(F^2)$ is invariant.
- It is sometimes convenient to choose a basis
$$T^a \in \mathrm{lie}(G) \quad ; \quad a = 1 \dots \dim(G).$$

Then $A_\mu = A_\mu^a T^a$; $F_{\mu\nu} = F_{\mu\nu}^a T^a$;

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

$\Downarrow$ decompose both sides in components & use the def. of "structure constants" f^{abc}:
$$[T^a, T^b] = i f^{abc} T^c$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$$

- Also: $\mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) = \frac{1}{2} F_{\mu\nu}^a F^{\mu\nu a}$ (using a canonical basis with $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$)

- The generalization to fermionic matter is straightforward:
$$\mathcal{L}_{matter} = \bar{\psi}(i\not{D} - m)\psi$$
$$= (\bar{\psi}_i)^a \left[i(\gamma^\mu)_a{}^b (\partial_\mu \delta^i_j + i A_\mu^a (T^a)^i{}_j) - m\, \delta_a^b \delta^i_j \right] (\psi^j)_b$$

(For a different repres., one simply replaces T^a by $R(T^a)$.)

13.2 The Standard Model

- The SM is a gauge theory with $G = G_{SM} = SU(3) \times SU(2) \times U(1)$

$$\Rightarrow \mathcal{L}_{gauge} = -\sum_{i=1}^{3} \frac{1}{2g_i^2} \operatorname{tr}\left(F_{\mu\nu}^{(i)} F^{(i)\mu\nu}\right)$$

$i = 3, 2, 1$

- Fermionic matter comes in 3 generations:

$$\mathcal{L}_{matter} = \sum_{a=1}^{3} \bar{\psi}_L^{\,a} \, i \not{D} \, \psi_L^{\,a} ,$$

(all left-handed spinors!)

where <u>for each a</u>, we have the fields

$$\psi_L^{\,a} = \{ \underbrace{Q^a}_{\binom{u}{d}}, (u^c)^a, (d^c)^a, \underbrace{L^a}_{\binom{\nu}{e}}, (e^c)^a \}$$

$$= (3,2)_{1/3} + (\bar{3},1)_{-4/3} + (\bar{3},1)_{2/3} + (1,2)_{-1} + (1,1)_2$$

F of SU(3)

F of SU(2)

charge under U(1)

(i.e. $\psi \to e^{i\alpha/3}\psi$)

compl. conj. of F of SU(3)

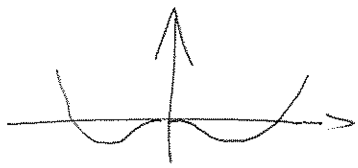
etc.

This line fixes all couplings in $\mathcal{L}_{matter}$

- There is also $\mathcal{L}_{Higgs}$ with $\phi = (1,2)_1$ and

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

wrong sign!

$\Rightarrow$ potential  forces $\langle \phi \rangle \neq 0$

$\Rightarrow$ mass for some of the gauge Bosons (W, Z)
from $\mathcal{L} \supset \sim A_\mu^{\,2} \phi^2$

$\Rightarrow$ mass for fermions from

$$\mathcal{L}_{Yukawa} \sim \underbrace{\phi \bar{\Psi} \Psi}_{\substack{\text{all possible terms which respect} \\ \text{gauge invariance ; different} \\ \text{"Yukawa couplings" fix masses.}}}$$