2 Free scalar field

2.1 Classical theory – Lagrangian formulation

- You (should) already know from electrodynamics:
$$S = \frac{1}{2}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
- Quantization of this theory is complicated by the fact that A_μ has 4 components & the action is gauge invariant ($A_\mu \to A_\mu + \partial_\mu \chi$).
- Hence, let's start with a toy model (which is, however, also relevant for Higgs & pions):
$$A_\mu(x) \longrightarrow \varphi(x)$$
- We formulate the theory in analogy to mechanics:

(1-dim.) mechanics	scalar FT
$x: t \mapsto x(t)$	$\varphi: x \to \varphi(x) \quad (x \in \mathbb{R}^4)$
$S = S[x] = \int dt\, L(x,\dot{x})$	$S = S[\varphi]$ $= \int dt\, L[\varphi(t,\bar{x}), \dot{\varphi}(t,\bar{x})]$

- Crucially, we assume that L is "local in x", i.e.
$$L = \int d^3\bar{x}\, \mathcal{L}(\varphi(x), \dot{\varphi}(x), \bar{\nabla}\varphi(x), \bar{\nabla}\dot{\varphi}(x), \ldots),$$
with only finitely many higher derivatives.
- We could say that we generalize the "locality of $S[x]$ in t to locality in $\mathbb{R}^{1,3}$.

- Next we define $V(\varphi) \equiv -\mathcal{L}(\varphi = \text{const.})$ and assume that V has a minimum at $\varphi = \varphi_0$. W.L.O.G. let $\varphi_0 = 0$ and $V(\varphi_0) = 0$, such that

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \dots$$

- Since we are first interested in small excitations of the ground state or vacuum $\varphi = 0$, we assume φ to be small, thus ignoring higher terms in the Taylor expansion for now.
- However, we should allow for some dependence on $\dot{\varphi}$. Clearly, to have a chance of Poinc. inv., we must allow for $\partial_\mu \varphi$ to appear ($\mu = 0..3$).
- Simple fact: $(\partial_\mu \varphi)(\partial_\nu \varphi)\eta^{\mu\nu}$ is the lowest-order invariant term (in powers of φ).

$$\Rightarrow \boxed{S = \int d^4x\, \mathcal{L} = \int d^4x \left(\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 \right)}$$

is the unique $O(\varphi^2)$ Poinc.-inv. action, which is also a (mostly) sensible approx. to the general, local, interacting theory.

- To gain some intuition, let's separate time & space ($\varphi(x) = \varphi(t, \bar{x})$) and discretize the latter:

$$\int d^3\bar{x} \longrightarrow \sum_{\bar{x} \in 3\,\text{dim. lattice}}$$

with $\bar{x} = \Delta \cdot (0,0,0)\,;\ \Delta \cdot (0,0,1)\,;\ \Delta \cdot (0,0,-1)$

Δ $\uparrow$ lattice spacing. ....

- This corresponds to

$$L = \int d^3\bar{x}\, \mathcal{L} = \int d^3\bar{x} \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\bar{\nabla}\varphi)^2 - \frac{m^2}{2}\varphi^2 \right)$$

$\downarrow$

$$L = \sum_{\bar{x}} \Big\{ \underbrace{\frac{1}{2}\dot{\varphi}(t,\bar{x})^2}_{T} \underbrace{- \frac{1}{2}\sum_{i=1}^{3} \left(\frac{\varphi(t, \bar{x} + \hat{e}_i \Delta) - \varphi(t,\bar{x})}{\Delta} \right)^2 - \frac{m^2}{2}\varphi^2}_{V_{\text{class. Mech.}}} \Big\}$$

$\Longrightarrow$ We have a "mech. system" with infinitely many degrees of freedom ($\varphi(t,\bar{x})\,;$ "$\bar{x} \in \mathbb{R}^3$").
The free dynamics is simply that of a set of coupled <u>harm. oscillators</u>. It will be easy to decouple them – see below.

- <u>Equation of motion</u>:

$$0 \stackrel{!}{=} \delta S = \delta \int d^4x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right)$$

$$= \int d^4x \left((\partial_\mu \varphi)\eta^{\mu\nu}(\partial_\nu \delta\varphi) - m^2 \varphi\,\delta\varphi \right)$$

$$= -\int d^4x \left(\eta^{\mu\nu}\partial_\mu\partial_\nu \varphi + m^2 \varphi \right) \delta\varphi$$

$$\Rightarrow \boxed{(\partial^2 + m^2)\varphi = 0} \quad \text{Klein-Gordon-equ.}$$

- Solutions are plane waves, e.g.

$$\varphi(x) = \varphi_0 \sin kx \quad \text{with} \quad k^2 - m^2 = 0.$$

- Here $k = (k^0, \bar{k})$ corresponds to the 4-momentum of a particle (see later). Choosing $k^0 = m, \bar{k} = 0$ corr. to a particle / particles at rest. m will turn out to be the particle mass.

2.2. Classical theory - Hamiltonian formulation

- Class. mech: $L(q_i, \dot{q}_i) \longrightarrow H(q_i, \pi_i) = \sum_i \pi_i \dot{q}_i - L$

$$\text{with} \quad \pi_i = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}$$

- FT on lattice: $L(\varphi, \dot{\varphi}) = \sum_{\bar{x}} \frac{1}{2} \dot{\varphi}(\bar{x})^2 + \dots$

plays the role of index "i"

$$\pi(\bar{x}) = \frac{\partial L}{\partial \dot{\varphi}(\bar{x})} = \dot{\varphi}(\bar{x})$$

$$H = \sum_{\bar{x}} \pi(\bar{x}) \dot{\varphi}(\bar{x}) - L$$

$$= \sum_{\bar{x}} \frac{1}{2} \pi(\bar{x})^2 - \dots$$

- Continuum FT:

- We need to generalize $\frac{\partial L}{\partial q_i}$ to the case where i becomes continuous, i.e., L is a functional.

Mathem. interlude: Let $F: f \mapsto \mathbb{R}$ be a functional. The functional derivative $\frac{\delta F}{\delta f(x)}$ is defined by

$$F[f+\varepsilon] - F[f] = \int dx \, \frac{\delta F}{\delta f(x)} \cdot \varepsilon(x) + O(\varepsilon^2)$$

- If we, in addition, use the natural generalization

$$\sum_i \pi_i \dot{q}_i \longrightarrow \int d^3x \, \pi(\bar{x}) \dot{\varphi}(\bar{x}),$$

then we arrive at the following continuum version of the lagrangian/hamiltonian transition:

$$\pi(\bar{x}) = \frac{\delta}{\delta \dot{\varphi}(\bar{x})} L[\varphi, \dot{\varphi}] \quad ; \quad H[\varphi, \pi] = \int d^3x \, \pi \dot{\varphi} - L$$

- In full generality, $F = \int dx \, A(f(x))$ implies $\frac{\delta F}{\delta f(x)} = \frac{\partial A}{\partial f}(x)$. [Check this if it's not obvious!]

- Hence, $\pi(\bar{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}(\bar{x}) = \dot{\varphi}(\bar{x})$ and

$$H[\varphi, \pi] = \int d^3x \, \pi \dot{\varphi} - L = \int d^3x \, \frac{1}{2} \left(\pi^2 + (\bar{\nabla} \varphi)^2 + m^2 \varphi^2 \right) = \ldots$$

$\ldots = \int d^3x \,\mathcal{H}$ with $\mathcal{H}$ the "hamiltonian density".

2.3 Quantization (canonical)

- The usual postulates:
  $(\hbar = c = 1)$

$$[q_i, \pi_j] = i\delta_{ij}$$
$$[q_i, q_j] = 0$$
$$[\pi_i, \pi_j] = 0$$

- Obvious generalization to our continuum case:

$$[\varphi(\bar{x}), \pi(\bar{y})] = i\delta^3(\bar{x}-\bar{y})$$
$$[\varphi(\bar{x}), \varphi(\bar{y})] = 0$$
$$[\pi(\bar{x}), \pi(\bar{y})] = 0$$

- With $H = \int d^3\bar{x}\, \frac{1}{2}\left(\pi^2 + (\bar{\nabla}\varphi)^2 + m^2\varphi^2\right)$ the similarity to a set of harm. oscillators is obvious.

- They are coupled due to "$\bar{\nabla}$" and can thus be easily decoupled by Fourier trf.:

$$\varphi(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\bar{p}\cdot\bar{x}} \tilde{\varphi}(\bar{p}) \,;\quad \tilde{\varphi}(\bar{p}) = \int d^3x\, e^{-i\bar{p}\cdot\bar{x}} \varphi(\bar{x})$$

(analogously $\pi(\bar{x}) \leftrightarrow \tilde{\pi}(\bar{p})$)

- The commutation relations for $\tilde{\varphi}, \tilde{\pi}$ read:

$$[\tilde{\varphi}(\bar{p}), \tilde{\pi}(\bar{q})] = \int d^3x\, d^3y\, e^{-i(\bar{p}\bar{x} + \bar{q}\bar{y})} [\varphi(\bar{x}), \pi(\bar{y})]$$

$$= i\int d^3x\, e^{-i(\bar{p}+\bar{q})\bar{x}} = i(2\pi)^3 \delta^3(\bar{p}+\bar{q})$$

$$[\tilde{\varphi}(\bar{p}), \tilde{\varphi}(\bar{q})] = [\tilde{\pi}(\bar{p}), \tilde{\pi}(\bar{q})] = 0 \quad \text{(obvious)}$$

- Of the 3 terms in H we only check the most interesting one:

$$\int d^3x\,(\bar\nabla\varphi)^2 = \int d^3x \int \frac{d^3p}{(2\pi)^3}(i\bar p)e^{i\bar p\bar x}\tilde\varphi(\bar p)\int\frac{d^3q}{(2\pi)^3}(i\bar q)\cdot e^{i\bar q\bar x}\tilde\varphi(\bar q) = \cdots$$

x-integration $\Rightarrow (2\pi)^3\delta^3(\bar p+\bar q)$

$$\cdots = \int\frac{d^3p}{(2\pi)^3}\,\bar p^2\tilde\varphi(\bar p)\tilde\varphi(-\bar p).$$

- Observe that $\varphi(\bar x)^+ = \varphi(\bar x) \Rightarrow \tilde\varphi^+(\bar p) = \tilde\varphi(-\bar p)$.
- Thus, including also the other two terms, we have:

$$H = \int\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2}\left(|\tilde\pi|^2 + (\bar p^2+m^2)|\tilde\varphi|^2\right)$$

to be read either literally or as $\tilde\pi\tilde\pi^+$, depending on whether we are before or after quantization.

- To make the analogy to a system of harm. oscillators even more perfect, let

$$\omega_{\bar p} \equiv \sqrt{\bar p^2+m^2}.$$

(oscillation frequency of classical wave solution, see above)

$$\Rightarrow H = \int\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2}\left(|\tilde\pi|^2+\omega_{\bar p}^2|\tilde\varphi|^2\right)$$

Reminder: $H = \frac{1}{2}(\pi^2+\omega^2q^2)$; $[q,\pi]=i$

$$\downarrow \quad a = \frac{1}{2}\left(\sqrt{2\omega}\,q + i\sqrt{\tfrac{2}{\omega}}\,\pi\right);\ a^+ = \cdots$$

$H = \omega\left(a^+a+\frac{1}{2}\right)$; $[a,a^+]=1$

- Motivated by this, let us make the ansatz

$$a_{\bar p} = \frac{1}{2}\left(\sqrt{2\omega_{\bar p}}\,\tilde\varphi(\bar p) + i\sqrt{\frac{2}{\omega_{\bar p}}}\,\tilde\pi(\bar p)\right)$$

$$a^+_{\bar p} = \frac{1}{2}\left(\sqrt{2\omega_{\bar p}}\,\tilde\varphi(-\bar p) - i\sqrt{\frac{2}{\omega_{\bar p}}}\,\tilde\pi(-\bar p)\right)$$

[Note that $\bar p \to -\bar p$!]

Comment: It is not yet totally clear that this will work since our analogy is not perfect: Unlike "p, q" our $\tilde\varphi, \tilde\pi$ are not real and they are not (quite) conjugate variables: $\delta^3(\bar p + \bar q) = \delta^3(\bar p - (-\bar q))$, i.e. $\tilde\varphi(\bar p)$ is conjugate to $\tilde\pi(-\bar p)$. We could keep "massaging" our class. system into perfect agreement with a set of oscillators and only then introduce a, a^+, but we wouldn't gain much new information.

- It is easy to derive the commutation relations:

$$[a_{\bar p}, a^+_{\bar q}] = (2\pi)^3\delta^3(\bar p - \bar q)$$

$$[a_{\bar p}, a_{\bar q}] = [a^+_{\bar p}, a^+_{\bar q}] = 0.$$

- Furthermore, we have

$$\tilde\varphi(\bar p) = \frac{1}{\sqrt{2\omega_{\bar p}}}\left(a_{\bar p} + a^+_{-\bar p}\right) \; ; \quad \tilde\pi(\bar p) = -i\sqrt{\frac{\omega_{\bar p}}{2}}\left(a_{\bar p} - a^+_{-\bar p}\right)$$

and hence

$$H = \int \frac{d^3\bar p}{(2\pi)^3}\cdot\frac{1}{2}\left\{\frac{\omega_{\bar p}}{2}\left(a_{\bar p} - a^+_{-\bar p}\right)\left(a^+_{\bar p} - a_{-\bar p}\right) + \omega^2_{\bar p}\frac{1}{2\omega_{\bar p}}\left(a_{\bar p} + a^+_{-\bar p}\right)\left(a^+_{\bar p} + a_{-\bar p}\right)\right\}$$

... "cross-terms" cancel ; can use $\bar p \to -\bar p$ freely ...

$$\Rightarrow H = \int \frac{d^3\bar{p}}{(2\pi)^3} \omega_{\bar{p}} \cdot \frac{1}{2} \left(a^+_{\bar{p}} a_{\bar{p}} + a_{\bar{p}} a^+_{\bar{p}} \right)$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \cdot \omega_{\bar{p}} \Big(a^+_{\bar{p}} a_{\bar{p}} + \frac{1}{2} \underbrace{[a_{\bar{p}}, a^+_{\bar{p}}]}_{(2\pi)^3 \delta^3(\bar{0}) \text{ "=" } \int d^3x\, e^{i\bar{0}\cdot\bar{x}} \text{ "=" } Vol(\mathbb{R}^3) = V} \Big)$$

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\bar{p}} a^+_{\bar{p}} a_{\bar{p}} + V \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2} \omega_{\bar{p}}$$

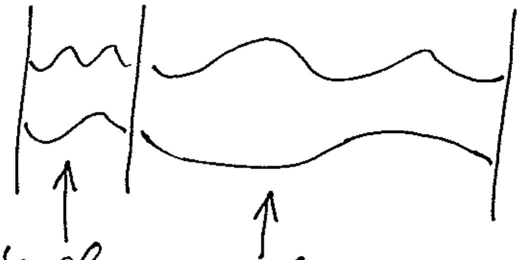
This is easy to derive "properly" using T^3 instead of $\mathbb{R}^3$, where $\delta^3(\bar{p}-\bar{q})$ becomes $\delta_{p^1 q^1} \delta_{p^2 q^2} \delta_{p^3 q^3}$.

This is truly divergent if no "UV-cutoff" is introduced.
$\Rightarrow$ The <u>vacuum energy density</u> diverges due to contributions of zero-point-energies of "oscillators" with arbitrarily high frequency

<u>Vacuum energy</u>:

- Irrelevant for QFT in $\mathbb{R}^3$ since it can be absorbed in overall constant shift of H.
- Relevant if QFT is coupled to gravity since it then curves space-time ("Cosm. constant problem").
- In non-trivial geometries (e.g. QED with conducting plates) the energies of the low-lying modes can be manipulated by moving the plates. This leads to a finite effect (force on the plates) independently

of the still present divergence at $|\bar{p}| \to \infty$:



p's / ω's change if middle plate is moved
($\to$ Casimir energy / Casimir effect.)

So far, we have found:

$$\varphi(\bar{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\bar{p}}}} e^{i\bar{p}\bar{x}} \left(a_{\bar{p}} + a^+_{-\bar{p}}\right) \; ; \quad \pi(\bar{x}) = \dots$$

with $\omega_{\bar{p}} = \sqrt{\bar{p}^2 + m^2}$; $\quad [a_{\bar{p}}, a^+_{\bar{q}}] = (2\pi)^3 \delta^3(\bar{p} - \bar{q})$

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\bar{p}} a^+_{\bar{p}} a_{\bar{p}} \qquad \text{(we discard the vac. energies)}$$

- At the moment, this is just an (operator) algebra with one distinguished operator: H.
- Physics starts if we also provide a Hilbert-space representation of this algebra
- We start by postulating a ("vacuum") state $|0\rangle$ which obeys $\quad a_{\bar{p}}|0\rangle = 0 \quad (\forall \bar{p}.)$
- Next we define "one-particle states" $\quad a^+_{\bar{p}}|0\rangle \quad (\forall \bar{p})$

It is easy to calculate the energy of these states:

$$H a^+_{\bar{p}}|0\rangle = \int \frac{d^3k}{(2\pi)^3} \omega_{\bar{k}} a^+_{\bar{k}} a_{\bar{k}} a^+_{\bar{p}} |0\rangle = \int \frac{d^3k}{(2\pi)^3} \omega_{\bar{k}} a^+_{\bar{k}} \delta^3(\bar{k} - \bar{p}) |0\rangle$$

$$\Rightarrow \quad H a^+_{\bar p}|0\rangle = \omega_{\bar p}\, a^+_{\bar p}|0\rangle$$

Since $\omega_{\bar p} = \sqrt{\bar p^2 + m^2}$ is the energy of a particle with mass m and momentum $\bar p$, the name of such states makes sense!

- <u>Two-particle states:</u> $\quad a^+_{\bar p} a^+_{\bar q}|0\rangle \quad (\forall \bar p, \bar q)$

  Easy to check: $\quad H a^+_{\bar p} a^+_{\bar q}|0\rangle = \underbrace{(\omega_{\bar p} + \omega_{\bar q})}_{\text{energy of two, non-interacting particles.}} a^+_{\bar p} a^+_{\bar q}|0\rangle$

  <u>Idea:</u> Commute $a_{\bar k}$, the annihilation operator, to the right until it hits the vacuum. Each time it passes one of the creation operators, pick up the δ^3-fct. and hence the $\omega_{\bar k}$.

- This extends to any number of particles and gives the <u>Fock space.</u>

- Choosing the vacuum normalization $\||0\rangle|^2 = \langle 0|0\rangle = 1$, we easily find

$$(a^+_{\bar p}|0\rangle)\cdot(a^+_{\bar q}|0\rangle) = \langle 0| a_{\bar p} a^+_{\bar q}|0\rangle = (2\pi)^3\,\delta^3(\bar p - \bar q)$$

(δ-fct. normalization, as in QM of free particle)

- It will be convenient to use the notation & normalization

$$|\bar p\rangle \equiv \sqrt{2\omega_{\bar p}}\, a^+_{\bar p}|0\rangle$$

$$|\bar p, \bar q\rangle \equiv \sqrt{2\omega_{\bar p}}\sqrt{2\omega_{\bar q}}\, a^+_{\bar p} a^+_{\bar q}|0\rangle \quad \text{etc.}$$

such that, e.g., $\langle \bar{p} | \bar{q} \rangle = 2\omega_{\bar{p}} (2\pi)^3 \delta^3(\bar{p}-\bar{q})$.

(The utility of the extra factor $\omega_{\bar{p}}$ will become clear later on. Some authors find it convenient to define $a'_{\bar{p}} \equiv \sqrt{2\omega_{\bar{p}}}\, a_{\bar{p}}$ such that $|p\rangle = a'^{\dagger}_{\bar{p}} |0\rangle$, but we won't do this. Be careful when comparing formulae from different books!)

2.4 Complex scalar

$$\mathcal{L} = \eta^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi^*) - m^2 \phi \phi^* = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

(sloppy but widely used notation)

- Note the different normalization conventions compared to the real scalar
- With $\phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$ this goes over into precisely twice the scalar lagrangian we already know.
- Nevertheless, it is useful to quantize this without giving up the complex notation:

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

- Treating ϕ, ϕ^* as independent, we get

$$-m^2 \phi^* - \Box \phi^* = 0 \qquad (\Box \equiv \partial_\mu \partial^\mu).$$

- Doing the same with ϕ^* (or just by complex conjugation) we also have: $(\Box + m^2)\phi = 0.$

- $\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi^* \; ; \quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot\phi^*} = \dot\phi$

$$\mathcal{H} = \pi\dot\phi + \pi^*\dot\phi^* - \mathcal{L} = |\pi|^2 + |\nabla\phi|^2 + m^2|\phi|^2$$

- <u>Quantization:</u> $[\phi(\bar{x}), \pi(\bar{y})] = [\phi^+(\bar{x}), \pi^+(\bar{y})] = i\delta^3(\bar{x}-\bar{y})$

- Recall that, for the real scalar, we were successful with the ansatz

$$\varphi(\bar{x}) = \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2\omega_{\bar{p}}}}\, e^{i\bar{p}\bar{x}}\left(a_{\bar{p}} + a^+_{-\bar{p}}\right) = \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2\omega_{\bar{p}}}}\left(a^+_{\bar{p}} e^{-i\bar{p}\bar{x}} + a_{\bar{p}} e^{i\bar{p}\bar{x}}\right)$$

- Reality was encoded in $a_{\bar{p}}$ & $a^+_{\bar{p}}$ being conjugate to each other. Let's hence try to generalize by

$$\phi(\bar{x}) = \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2\omega_{\bar{p}}}}\left(a^+_{\bar{p}} e^{-i\bar{p}\bar{x}} + b_{\bar{p}} e^{i\bar{p}\bar{x}}\right)$$

- One finds: The ϕ, π, ϕ^*, π^* commut. relations imply that $a_{\bar{p}}, a^+_{\bar{p}}$ & $b_{\bar{p}}, b^+_{\bar{p}}$ represent two sets of independent oscillators.

- $H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\bar{p}} \left(a^+_{\bar{p}} a_{\bar{p}} + b^+_{\bar{p}} b_{\bar{p}}\right)$

(As we will see, this model has a conserved charge, due to its symm. $\phi \to e^{i\alpha}\phi$. "a/b-particles" are pos./neg. charged.)