3 Noether Theorem

3.1 Formulation and derivation in field theory

With every continuous symmetry of the action comes a conserved current density (and hence a cons. charge)

Note: This is very similar to the Noether theorem of mechanics. The crucial novelty is the current.

Derivation: • By assumption, our symm. is continuous and hence an infinitesimal symm. trf. can be defined:

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \varepsilon X(x).$$

• This modification $\delta_\epsilon \varphi = \varphi' - \varphi$ of the field induces a modification of $\delta_\epsilon \mathcal{L} = \mathcal{L}' - \mathcal{L} = \mathcal{L}(\varphi', \partial\varphi') - \mathcal{L}(\varphi, \partial\varphi)$ of the lagrangian. We specify the term symmetry by demanding

$$\boxed{\delta_\epsilon \mathcal{L} = \epsilon\, \partial_\mu F^\mu(x)}\,.$$

(In words: $\mathcal{L}$ changes only by the divergence of some appropriate vector field F^μ.)

• Indeed, assume that φ satisfies the EOM and check whether φ' does so too: We need to check that $\delta S' = 0$ for any variation $\delta\varphi$ in a bounded region.

But $\delta S' = \delta(\delta_\epsilon S) + \delta S = \delta(\delta_\epsilon S) =$

$$= \int d^4x\, \delta(\delta_\epsilon \mathcal{L}) = \int d^4x\, \epsilon\, \partial_\mu \delta F^\mu(x) = 0$$

by Gauss' law since $\delta\varphi$ and hence δF vanish outside a bounded region.

- <u>Note</u>: We did <u>not</u> demand that $\delta_\epsilon S = 0$ or that $\delta_\epsilon \varphi$ should vanish outside a bounded region.
- Now a simple calculation leads to the result:

$$\epsilon \partial_\mu F^\mu = \delta_\epsilon \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi} \delta_\epsilon \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_\epsilon \partial_\mu \varphi$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi} \delta_\epsilon \varphi + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_\epsilon \varphi \right] - \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] \delta_\epsilon \varphi .$$

Use the EOM $\quad \dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0 \quad$ and $\delta_\epsilon \varphi = \epsilon X$

to obtain:

$$\partial_\mu F^\mu = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} X \right]$$

or

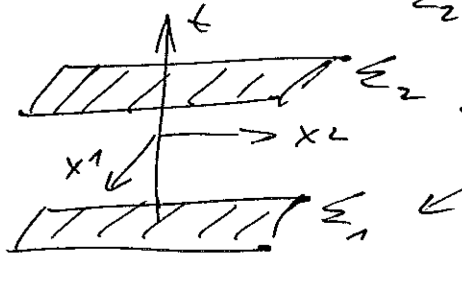
$$\boxed{ j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \cdot X - F^\mu \quad \text{is conserved}, \quad \partial_\mu j^\mu = 0 }$$

- For many field configurations (in particular if they are spatially bounded) j^μ will fall off sufficiently fast at $|\vec{x}| \to \infty$. Then

$$\boxed{ Q(t) \equiv \int d^3x \, j^0(t, \vec{x}) }$$

will be finite and defines a <u>conserved charge</u>:

$$Q(t_2) - Q(t_1) = \int_{\Sigma_2} df_\mu j^\mu - \int_{\Sigma_1} df_\mu j^\mu = \int_{\text{Vol.}} \partial_\mu j^\mu = 0$$



two infinite spatial slices.

<u>Problems:</u> • Prove $\dot{Q}=0$ directly, i.e. without first introducing a finite $\Delta t = t_2 - t_1$.

• Unless you find it obvious, apply the above derivation of Noether's theorem to mechanics and demonstrate, e.g., energy conservation for any system without explicit time-dependence in L.

3.2 Energy-momentum conservation

• The relevant symm. are translations in $\mathbb{R}^{1,3}$:

$$x^\mu \longrightarrow x'^\mu = x^\mu - \varepsilon^\mu \quad (\text{transl. by } "-\varepsilon")$$

(4 symms., hence a "4-vector of ε's")

• $\varphi'(x) = \varphi(x+\varepsilon)$; $\delta_\epsilon \varphi = \varphi(x+\epsilon) - \varphi(x) = \epsilon^\nu \partial_\nu \varphi$

$\mathcal{L}'(x) = \mathcal{L}(x+\varepsilon)$ [We do not even need to know the specific way in which $\mathcal{L}$ depends on φ, $\partial\varphi$ and hence on x]

$$\Rightarrow \delta_\epsilon \mathcal{L} = \mathcal{L}(x+\varepsilon) - \mathcal{L}(x) \approx \epsilon^\mu \partial_\mu \mathcal{L} = \epsilon^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L})$$

$$\equiv \underbrace{\epsilon^\nu \partial_\mu (F^\mu{}_\nu)}$$

linear superposition of 4 contributions of type $\epsilon \partial_\mu F^\mu$, labelled by $\nu = 0 \cdots 3$.

$$\boxed{F^\mu{}_\nu \equiv \delta^\mu{}_\nu \mathcal{L}}$$

$$\boxed{X_\nu \equiv \partial_\nu \varphi}$$

- We apply our general formula four times, getting four conserved currents, labelled by ν:

$$j^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} X_\nu - F^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L}$$

- We have found that the "<u>energy-momentum tensor</u>"

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}}$$

is conserved, $\partial_\mu T^{\mu\nu} = 0$.

- The name is justified by the observation

$$\int d^3x \, T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L} \right) = H = P^0$$

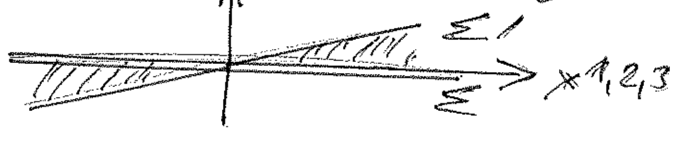
1st component of the 4-vector $\{P^\mu\}$ of energy-mom. of the full system.

- More generally:

$$P^\nu = \int d^3x \, T^{0\nu}$$

(Note that $\{P^\nu\}$ is a 4-vector in spite of its apparently non-covariant definition. In particular P^ν does not change if the space-like hyperplane used in its def. is rotated.)

$$P_\nu = \int d^3x \, T^{0\nu} = \int_\Sigma df_\mu \, T^{\mu\nu} = \int_{\Sigma'} df_\mu \, T^{\mu\nu}$$



- It is easy to work out:

$$P^i = \int d^3x\, T^{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot\varphi} \partial^i \varphi = -\int d^3x\, \pi (\nabla \varphi)_i$$

- Rewriting π, φ in terms of a, a^+ and emphasizing the operator-nature of P by "$\wedge$", we have

$$\hat{P}^i = \int \frac{d^3q}{(2\pi)^3} q^i \underbrace{a^+_{\bar q} a_{\bar q}}_{\text{also known as "particle-number operator".}} \quad \& \quad \hat{P}^\mu |p\rangle = p^\mu |p\rangle$$

- We can also perform the above analysis for a complex scalar:

$$\hat{P}^\mu = \int \frac{d^3q}{(2\pi)^3} q^\mu \left(a^+_{\bar q} a_{\bar q} + b^+_{\bar q} b_{\bar q} \right)$$

here $q^0 \equiv \sqrt{\bar q^2 + m^2}$.

- With this, the particle interpretation of our Fock-space is fully justified.

<u>Important comment:</u>

- From its def., our "canonical" $T^{\mu\nu}$ need not be symmetric
- While it happens to be symmetric for the real scalar (explicitly, $T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}$), this fails already in QED.
- However, $T^{\mu\nu}$ can always be made symmetric by adding an independently conserved current which also does not modify P^ν.

- In GR, one uses the def.

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{-\det(g_{\delta\sigma})}} \cdot \frac{\delta S}{\delta g_{\mu\nu}(x)}$$

(where S is formulated with a general metric $g_{\mu\nu}$ rather than $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$).

- This directly gives the symmetric form of $T^{\mu\nu}$, which in fact is very important because, e.g.

$$\mathcal{L} \supset h_{\mu\nu} T^{\mu\nu}$$

(with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) characterizes the coupling of a QFT to gravity.

3.3 U(1)-symmetry and charge of the compl. scalar

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

U(1)-Symmetry: $\phi \longrightarrow \phi' = e^{i\varepsilon}\phi = \phi + i\varepsilon\phi + \cdots$

$\phi^* \longrightarrow \phi^{*\prime} = e^{-i\varepsilon}\phi^* = \phi^* - i\varepsilon\phi^* + \cdots$

Note: As before, we treat ϕ, ϕ^* as indep. fields during the calculation. This is justified by truly enlarging our field space from 2 to 4 real dims.: $\mathcal{L} = \eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\psi + \cdots$, $\phi, \psi \in \mathbb{C}$ and projecting on the real subspace, $\psi = \phi^*$, at the very end.

- Our formula $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} X - F^\mu$ has the obvious multi-field generalization

$$j^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} X^i - F^\mu .$$

- In our case $\mathcal{L} = \mathcal{L}' \Rightarrow F^\mu = 0$ and $\{\varphi^i\} = \{\phi, \phi^*\}$.

$$\Rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} X + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} X^* \quad ; \quad X = i\phi \; ; \; X^* = -i\phi^*$$

$$j^\mu = (\partial^\mu \phi^*) i\phi + (\partial^\mu \phi)(-i\phi^*) = \underline{\underline{-i(\phi^* \overleftrightarrow{\partial^\mu} \phi)}}$$

(Common notation: $A \overleftrightarrow{\partial^\mu} B \equiv A \partial^\mu B - (\partial^\mu A) B$)

- $Q = \int d^3x \, j^0 = -i \int d^3x \, \phi^+ \overleftrightarrow{\partial_t} \phi = \int \frac{d^3p}{(2\pi)^3} \left(a^+_{\bar{p}} a_{\bar{p}} - b^+_{\bar{p}} b_{\bar{p}} \right)$

Check this!

$\Rightarrow$ We can think of the states created by a^+ / b^+ as particles / antiparticles with same mass but opposite charge.