4 Heisenberg picture, Causality and Covariance

4.1 Heisenberg picture

- Everything we did so far was in the Schrödinger picture:
  - Our operators $\varphi(\bar{x})$, $\pi(\bar{x})$ carry no time-depend. (We can think of them as emerging from quantization at $t=0$, although we did not even display the time-argument.)
  - Our states $|0\rangle$, $\sqrt{2\omega_{\bar{p}}}\, a_{\bar{p}}|0\rangle = |p\rangle$ etc. would have to evolve in time in the standard way,

$$|p\rangle \equiv |p_0\rangle \quad ; \quad |p_t\rangle = \exp(-iHt)|p_0\rangle ,$$

    to describe dynamics.

- This is not natural for a Poincare-inv. theory since the t-dependence sits in the states while the $\bar{x}$-dependence sits in the operators (e.g. $\varphi(\bar{x})$).

- Thus, we now change from

$$\text{Schrödinger picture} \quad - \quad O \text{ fix} \; ; \; |\psi_t\rangle = e^{-iHt}|\psi\rangle$$

to

$$\text{Heisenberg picture} \quad - \quad O = O_t \; ; \; |\psi\rangle \text{ fix} .$$

- The physical requirement of unchanged measurements

$$\langle \psi_t | O | \psi_t \rangle = \langle \psi | O_t | \psi \rangle$$

implies

$$\boxed{O_t = e^{iHt}\, O\, e^{-iHt}}$$

- For us, the most interesting time-dep. operator is

$$\varphi(x) = \varphi(t,\bar{x}) = e^{iHt} \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\bar{p}}}} \left(a_{\bar{p}}\, e^{i\bar{p}\bar{x}} + a^+_{\bar{p}}\, e^{-i\bar{p}\bar{x}} \right) e^{-iHt} .$$

- To simplify this, we need be able to "commute H with a, a^+". Using the explicit formula for H & a/a^+-commut. relations, it is easy to check

$$H a_{\bar{p}} = a_{\bar{p}} (H - \omega_{\bar{p}})$$
$$H a^+_{\bar{p}} = a^+_{\bar{p}} (H + \omega_{\bar{p}}).$$

- Note: This is in fact obvious without any calculation since, for $|\psi\rangle$ with $H|\psi\rangle = E|\psi\rangle$:

$$H a^+_{\bar{p}} |\psi\rangle = \underset{\substack{\uparrow \\ \text{since } a^+ \text{ creates a particle}}}{(E + \omega_{\bar{p}})} a^+_{\bar{p}} |\psi\rangle = a^+_{\bar{p}} (H + \omega_{\bar{p}}) |\psi\rangle$$

- Since an exponential of an operator is defined, e.g., by the Taylor series, we immediately see

$$e^{iHt} a_{\bar{p}} = a_{\bar{p}} e^{i(H - \omega_{\bar{p}})t} \quad ; \quad e^{iHt} a^+_{\bar{p}} = a^+_{p} e^{i(H + \omega_{\bar{p}})t}$$

- Since $p = \{p^0, \bar{p}\} = \{\omega_{\bar{p}}, \bar{p}\}$ for a particle "on the mass-shell", and since $px = p^0 x^0 - \bar{p}\bar{x}$, we find:

$$\underline{\underline{\varphi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2p^0}} \left(a_{\bar{p}} e^{-ipx} + a^+_{\bar{p}} e^{ipx} \right)}}$$

Note: – Defining $\pi(x) = \pi(t, \bar{x})$ analogously, one finds that $\pi(x) = \dot{\varphi}(x)$ at the operator level.

– Also, $(\Box + m^2)\varphi(x) = 0$ holds for the operator φ.

4.2 Causality

$\varphi(x)$ is also our simplest observable and, if "locality" still holds after quantization, we expect that measurements of φ at x & y do not interfere (do not influence each other) if $(x-y)^2<0$:

$$\boxed{[\varphi(x), \varphi(y)] = 0 \quad \text{for } (x-y)^2 < 0}$$

("Causality")

This is easy to check:

$$[\varphi(x), \varphi(y)] = \int \frac{d^3p}{(2\pi)^3\sqrt{2p^0}} \int \frac{d^3q}{(2\pi)^3\sqrt{2q^0}} \Big\{ [a_{\vec{p}}, a^+_{\vec{q}}]\, e^{-ipx+iqy}$$

$$+ [a^+_{\vec{p}}, a_{\vec{q}}]\, e^{ipx-iqy} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2p^0}\, e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3\, 2p^0}\, e^{ip(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\Big|_{p^0>0} e^{-ip(x-y)} - \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\Big|_{p^0>0} e^{ip(x-y)}$$

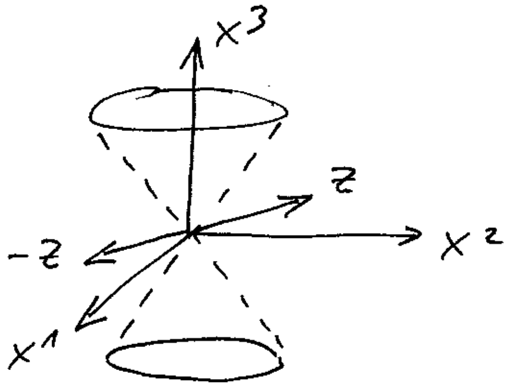
<u>Recall:</u> $\int dx\, \delta(f(x)) = \frac{1}{|f'(x_0)|}$; in our case $f(p^0) = (p^0)^2 - \omega_{\vec{p}}^2$

Each of the two terms is manifestly invariant under $SO^+(1,3)$.

<u>Fact:</u> Any vector z with $z^2<0$ can be transformed to $-z$ by $\Lambda \in SO^+(1,3)$. [cf. Problems]

Illustration:

(for $SO(1,2)$)



Thus, we can replace $(x-y)$ by $-(x-y)$ in, e.g., the second term without changing its value. Hence the terms cancel.

4.3 Covariance

- Of course, if $x^0 = y^0$, the claim $[\varphi(x), \varphi(y)] = 0$ immediately follows from canonical quantization. It is then tempting to claim the same for generic $x-y$ with $(x-y)^2 < 0$ on the basis of Lorentz / Poinc. covariance of the theory. However, we have broken this symm. in its manifest form during quantization and <u>not yet</u> established how it re-appears at the quantum level.

- We know that $\hat{H}, \hat{\bar{P}}$ generate translations in t and $\bar{x}$ in QM. We already have these operators in QFT and hence know that

$$\exp(i\hat{P}^\mu \varepsilon_\mu)$$

generates translations on states $|\psi\rangle$.

[Think carefully about the sign in the exponent!]

- We could extend our Noether-section to infinites.

Lorentz trfs.

$$\Lambda^\mu{}_\nu = \exp\left(i\varepsilon^{\rho\sigma}M_{\rho\sigma}\right)^\mu{}_\nu$$

$$= \delta^\mu{}_\nu + i\varepsilon^{\rho\sigma}(M_{\rho\sigma})^\mu{}_\nu + \dots$$

and construct the conserved charges coming with the continuous symms. parametrized by the $\varepsilon^{\rho\sigma}$. This would give us operators corresponding to $M_{\rho\sigma}$: $\hat{M}_{\rho\sigma}$.

- Now $\hat{\Lambda} = \exp\left(i t^{\rho\sigma}\hat{M}_{\rho\sigma}\right)$

  $\hat{T} = \exp\left(i d^\mu \hat{P}_\mu\right)$

  are operators realizing Lorentz-trfs. & translations on the Hilbert space. (In particular, the classical commut. relations of the symm. generators $M_{\rho\sigma}$ & P_μ apply to the operators $\hat{M}_{\rho\sigma}$ and $\hat{P}_\mu$.)

- In QM, after having realized our symm. group by unitary operators acting on the Hilbert space, we just need to check that they commute with H and we are done.

- Here, things are slightly different since, of course, boosts $\hat{\Lambda}$ do not commute with H. (They change the energy of states.)

- What we can and should demand is <u>covariance</u>

in the following sense:

- let us first measure the field value at a position $x \in \mathbb{R}^{1,3}$ in a state $|\psi\rangle$:

$$\langle\psi|\varphi(x)|\psi\rangle.$$

- Now, if we "rotate" our state by Λ and measure at the rotated position $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, we must clearly find the same result:

$$\langle\psi|\varphi(x)|\psi\rangle = \left(\langle\psi|\hat{\Lambda}^+\right)\varphi(\Lambda x)\left(\hat{\Lambda}|\psi\rangle\right).$$

(We drop the translation part for notational simplicity.)

- Thus, we need to demand

$$\boxed{\hat{\Lambda}\varphi(x)\hat{\Lambda}^+ = \varphi(\Lambda x)}.$$

- This can indeed be checked by expressing $\hat{M}_{SS}$ in a, a^+ and using commut. relations. (The finite version follows "by exponentiation".) It is of course also true for translations. [→ Problems]

- It is useful to check consistency of the above with the "group law" and also comare with the somewhat different relations for classical fields:

① <u>Field operators</u>

1st trf.: $\hat{\Lambda}_1\varphi(x)\hat{\Lambda}_1^+ = \varphi(\Lambda_1 x)$

2nd trf.: $\hat{\Lambda}_2 \hat{\Lambda}_1 \varphi(x) \hat{\Lambda}_1^+ \hat{\Lambda}_2^+ = \hat{\Lambda}_2 \varphi(\Lambda_1 x) \hat{\Lambda}_2^+$

$$= \varphi(\Lambda_2 \Lambda_1 x) .$$

This is OK since we have simply recovered the original relation with $\Lambda = \Lambda_2 \Lambda_1$.

② Classical fields

1st trf.: $(\Lambda_1 \varphi)(x) = \varphi(\Lambda_1^{-1} x)$

[Recall that $\Lambda_1 \varphi = \varphi'$ is the class. field configuration rotated by Λ_1.]

2nd trf.: $(\Lambda_2 (\Lambda_1 \varphi))(x) = (\Lambda_1 \varphi)(\Lambda_2^{-1} x)$

$$= \varphi(\Lambda_1^{-1} \Lambda_2^{-1} x) = \varphi((\Lambda_2 \Lambda_1)^{-1} x).$$

Also OK, but note the difference and don't get confused!

- Finally, we would also like to have covariance of our Fock-space basis:

$$\hat{\Lambda} |p\rangle = |p'\rangle \quad \text{with } p'^\mu = \Lambda^\mu{}_\nu p^\nu .$$

Due to the normalization ambiguity (our states are only δ-fct.-normalized and the prefactor of $\delta^3(\bar{p}-\bar{q})$ is, in principle, arbitrary) we can't a priori be sure about the prefactor in this relation.

- One can of course work this out explicitly using $a, a^\dagger$. [→ Problems]

- Here, we will merely do a consistency check to demonstrate that <u>our</u> normalization convention fits the "naive" relation above:

$$\langle p'|q'\rangle = \langle p|\hat{\Lambda}^\dagger\hat{\Lambda}|q\rangle = \langle p|q\rangle \quad (p' = \Lambda p;\ q' = \Lambda q)$$

$$2p_0'(2\pi)^3\delta^3(\bar{p}'-\bar{q}') = 2p_0(2\pi)^3\delta^3(\bar{p}-\bar{q}) \qquad (p_0 \equiv \omega_{\bar{p}} \text{ etc.})$$

It is obvious that both sides are non-zero only at the same point. So we just need to check normalization, which we can do e.g. by integrating with an arbitrary smooth fct. in $\bar{p}$. We choose $\int \frac{d^3p}{2p^0} = \int d^4p\,\delta(p^2-m^2)$.

- Now, by manifest covariance & $|\det\Lambda| = 1$:

<u>LHS:</u> $\int \frac{d^3p}{2p^0} 2p_0'\delta^3(\bar{p}'-\bar{q}') = \int d^4p\,\delta(p^2-m^2)2p_0'\delta^3(\bar{p}'-\bar{q}')$

$$= \int d^4p'\,\delta(p'^2-m^2)2p_0'\delta^3(\bar{p}'-\bar{q}') = \int \frac{d^3p'}{2p^{0\prime}} 2p_0'\delta^3(\bar{p}'-\bar{q}') = 1$$

<u>RHS:</u> $\int \frac{d^3p}{2p^0} 2p^0\delta^3(\bar{p}-\bar{q}) = 1$

<u>Note:</u> Cf. in particular the books by Itzykson/Zuber (Sect. 3-1-2) and Weinberg (Vol. I, Sect. 2).