5 Perturbation Theory – Leading Order Approach

5.1 The S-Matrix

- We have justified $V(\varphi) = \frac{m^2}{2}\varphi^2$ by Taylor exp.
- It is convenient to impose the discrete symm. $\varphi \to -\varphi$, such that $\lambda/4! \cdot \varphi^4$ is the next term

$$\Rightarrow \quad \mathcal{L}_0 \longrightarrow \mathcal{L}_0 + \mathcal{L}_{int} \quad ; \quad \mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$$
$$\mathcal{H}_0 \longrightarrow \mathcal{H}_0 + \mathcal{H}_{int} \quad ; \quad \mathcal{H}_{int} = +\frac{\lambda}{4!}\varphi^4$$

("$L = T - V$; $H = T + V$")

Comment:

- We could also drop the symm. requirement $\varphi \to -\varphi$ and consider $V_{int} = \frac{\lambda}{3!}\varphi^3$. This is sometimes dismissed because of instability at $\varphi \to -\infty$, but pert. theory around the local min. at $\varphi = 0$ is still ok. The real reason for starting with φ^4 is that it is slightly easier to deal with in our context.

- We will use the "Interaction Picture", i.e. keep the free time evolution of operators of the Heisenberg picture and let the additional time evolution act on states:

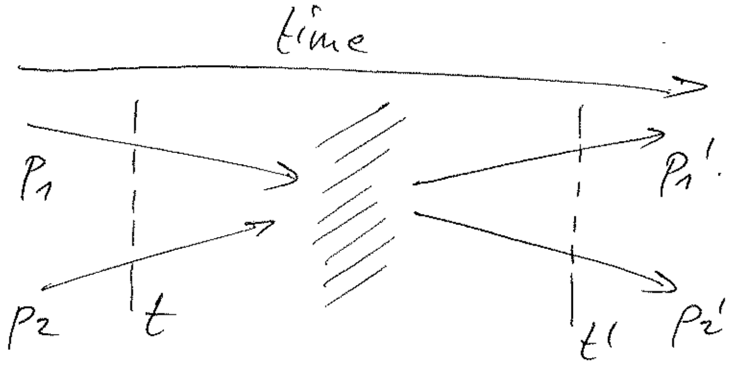
Schrödinger	$\longrightarrow$	Interaction ("I")
$\mathcal{O}$		$\mathcal{O}_t^I = e^{iH_0 t}\mathcal{O}e^{-iH_0 t}$
$\lvert\psi_t\rangle = e^{-iHt}\lvert\psi\rangle$		$\lvert\psi_t^I\rangle = e^{iH_0 t}e^{-iHt}\lvert\psi\rangle$

- Evolution from $t=0$ to t' is then analogously given by
$$|\psi^I_{t'}\rangle = e^{iH_0t'}e^{-iHt'}|\psi\rangle .$$

- Thus:
$$\boxed{\begin{array}{l}|\psi^I_{t'}\rangle = U(t',t)\,|\psi^I_t\rangle \\[2ex] \text{where } U(t',t) = e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}\end{array}}$$

(The operator U, which you should know from time-dep. pert. th. in QM, describes the unitary evolution of states in the interaction picture.)

- Since $\int d^3x\,\varphi^4 \supset a^+_{\bar p_1'}a^+_{\bar p_2'}a_{\bar p_1}a_{\bar p_2}$, we expect that at LO our interaction will induce <u>2-2-scattering</u> (two-to-two ...)



<u>Crucial:</u> We assume that, at t and t', the two particles are separated in space (wave packets rather than plane waves)

(<u>Note</u>: At higher orders in pert. th. or with more complicated interaction terms we could also have 2-n-scattering ($n>2$).)

- To work this out, let us first split the time evolution from t to t' into n small steps: $\Delta = \dfrac{t'-t}{n}$:

$$U(t',t) = U(t',t'-\Delta)\,U(t'-\Delta,t'-2\Delta)\cdots U(t+\Delta,t)$$

- Furthermore,

$$U(t+\Delta,t) = e^{iH_0(t+\Delta)}e^{-iH\Delta}e^{-iH_0t}$$

$$= e^{iH_0t}e^{iH_0\Delta}e^{-iH\Delta}e^{-iH_0t} = e^{iH_0t}e^{-iH_{int}}e^{-iH_0t} + O(\Delta^2)$$

$$= e^{-iH^I_{int}(t)\cdot\Delta} + O(\Delta^2),$$

where $H^I_{int}(t) = e^{iH_0t}H_{int}e^{-iH_0t}$ is the interaction Hamiltonian H_{int} transformed to the interaction picture.

- Thus,

$$U(t',t) = e^{-iH^I_{int}(t'-\Delta)\cdot\Delta}\,e^{-iH^I_{int}(t'-2\Delta)\cdot\Delta}\cdots e^{-iH^I_{int}(t)\cdot\Delta}$$

$$\equiv T\exp\left[-i\int_t^{t'} d\tau\, H^I_{int}(\tau)\right]$$

time-ordering "operator" (better: "symbol"), defined by

$$T\varphi(t_1)\varphi(t_2) = \begin{cases}\varphi(t_1)\varphi(t_2) & \text{if } t_1 \geq t_2\\ \varphi(t_2)\varphi(t_1) & \text{if } t_1 < t_2\end{cases}.$$

- Explicitly, we have

$$H^I_{int}(t) = e^{iH_0t}\int d^3x\,\frac{\lambda}{4!}(\varphi^I(x))^4 e^{-iH_0t} = \int d^3x\,\frac{\lambda}{4!}(\varphi^I(x))^4$$

interaction-picture field
$\equiv$ Heis.-pict. field of free theory

- We now define the <u>S-matrix</u>:

$$S = \lim_{\substack{t\to-\infty \\ t'\to+\infty}} U(t',t) = T e^{-i\int dt H^I_{int}(t)} = \underline{\underline{T e^{i\int d^4x \mathcal{L}^I_{int}(x)}}}$$

and the <u>S-matrix-element</u>:

$$\underline{\underline{S_{fi} = \langle p_1' p_2' | T \exp\left(i\int d^4x \mathcal{L}^I_{int}\right) | p_1 p_2 \rangle}}$$

(f/i stands for final/initial)

- Furthermore, we define the transition-matrix or T-matrix (do not confuse with time ordering!) by

$$S = \mathbb{1} - iT \quad \& \quad S_{fi} = \delta_{fi} + iT_{fi}$$

- To work this out, it will be convenient to change a/a^+-normalization such as to emphasize covariance rather than the harmonic-oscillator analogy:

$$(a_{\vec p})_{new} \equiv \sqrt{2\omega_{\vec p}}\,(a_{\vec p})_{old} \quad ; \quad \omega_{\vec p} = p^0$$

i.e.

$$\boxed{\begin{gathered}[a_{\vec p}, a^+_{\vec q}] = 2p^0 (2\pi)^3 \delta^3(\vec p - \vec q) \quad ; \quad |p\rangle = a^+_{\vec p}|0\rangle \\ \varphi^I(x) = \int \frac{d^3p}{(2\pi)^3 2p^0}\left(a_{\vec p}\, e^{-ipx} + a^+_{\vec p}\, e^{ipx}\right)\end{gathered}}$$

- At LO in λ we then have

$$iT_{fi} = \langle 0 | a_{\vec p_1'} a_{\vec p_2'} \left(-\frac{i\lambda}{4!}\right) \int d^4x \left(\varphi^I(x)\right)^4 a^+_{\vec p_1} a^+_{\vec p_2} |0\rangle$$

- The crucial point of all the previous manipulations was the expression of the matrix element in <u>free</u> fields. Based on the last 3 lines, an easy "commutator-style" calculation gives (see problems):

$$i\,T_{fi} = -i\lambda\,(2\pi)^4\,\delta^4(p_1+p_2-p_1'-p_2').$$

- Since the above "momentum-conservation-δ-fct." always arises in such situations, we define the "invariant matrix elements" $\mathcal{M}_{fi}$ by

$$\boxed{S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(\dots)\,\mathcal{M}_{fi}}$$

where, for 2-2-scattering in "$\lambda\varphi^4$-theory" we found:

$$i\mathcal{M}_{fi} = \underline{\underline{-i\lambda}}$$

- This is our first "Feynman rule":

$= -i\lambda$

(We can now "guess", why $\lambda\varphi^3$-theory would be slightly more complicated:


$\sim \lambda^2$.)

- Not surprisingly (due to the limit $t \to -\infty$ / $t' \to +\infty$), our result is singular ($\delta^4(..)$). The corresponding well-defined and finite quantity is the ...

5.2 Scattering cross section

- Consider a "fixed target experiment" with the beam (consisting of N_B particles of type B) is spread out over a transverse area F and hits a target consisting of one particle of type A:

$$\frac{N_{events}}{N_B} = \frac{\sigma}{F} \quad \Rightarrow \quad \sigma = \frac{N_{events}}{(N_B/F)}$$

transverse beam density

- We will need localized states and hence wave packets:

$$|f_{\bar p}\rangle \equiv \int d\tilde{k}\, f_{\bar p}(\bar k)\,|k\rangle \quad \text{where } d\tilde{k} \equiv \frac{d^3k}{(2\pi)^3\, 2k^0} \quad \text{and}$$

$f_{\bar p}(\bar k)$ is peaked near $\bar k = \bar p$, e.g.

$$f_{\bar p}(\bar k) \sim \exp(-\alpha|\bar k - \bar p|^2)$$

- Normalization:

$$\langle f_{\bar p}|f_{\bar p}\rangle = \int d\tilde{k} \int d\tilde{k}'\, f^*_{\bar p}(\bar k)\, f_{\bar p}(\bar k')\,\langle k'|k\rangle$$

$$= \int d\tilde{k}\, |f_{\bar p}(\bar k)|^2 \overset{!}{=} 1$$

- Next, check that $|f_{\bar p}\rangle$ is indeed localized in $\mathbb{R}^3$ for an appropriate choice of the function $f_{\bar p}$:

Indeed, using the by now very familiar expression for $\varphi^I(x)$ in terms of a, a^+, one easily checks that

$$a^+_{\bar{k}} = -i\int d^3x\, e^{-ikx} \overleftrightarrow{\partial_0} \varphi^I(x)\Big|_{x^0=0} .$$

Thus
$$|f_{\bar{p}}\rangle = \int d\tilde{k}\, f_{\bar{p}}(\bar{k})\, a^+_{\bar{k}}|0\rangle$$

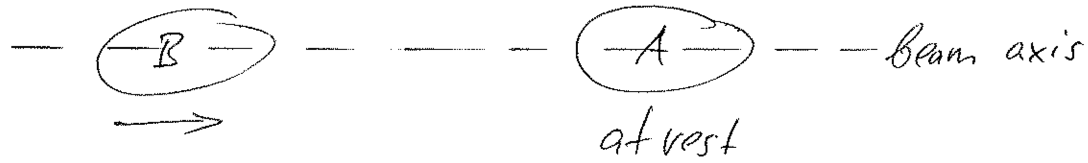
$$= \int d^3x \left\{ \int d\tilde{k}\, e^{i\bar{k}\bar{x}} \left(k_0 \varphi^I(\bar{x}) - i\dot{\varphi}^I(\bar{x})\right) f_{\bar{p}}(\bar{k}) \right\} |0\rangle$$

Now, with $f_{\bar{p}}(\bar{k})$ & $k_0 f_{\bar{p}}(\bar{k})$ smooth and localized in $\bar{k}$, their Fourier transforms

$$\int d\tilde{k}\, e^{i\bar{k}\bar{x}} f_{\bar{p}}(\bar{k}) \quad , \quad \text{—''— } \cdot k_0 \quad ,$$

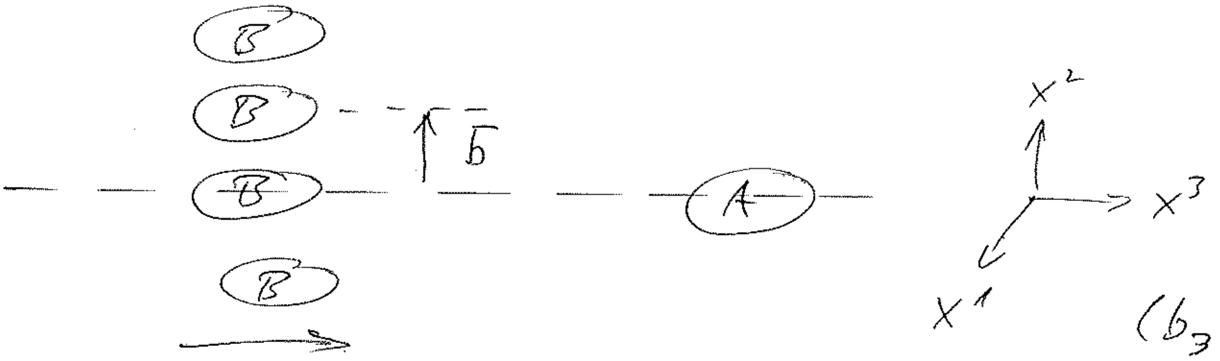
will be localized in $\bar{x}$. $|f_{\bar{p}}\rangle$ is then created by $\varphi^I(\bar{x})$ & $\dot{\varphi}^I(\bar{x})$ acting on $|0\rangle$ in a localized region near $\bar{x} = 0$.

- Thus, the state $|i\rangle = \int d\tilde{k}_A\, d\tilde{k}_B\, f_{\bar{p}_A}(\bar{k}_A) f_{\bar{p}_B}(\bar{k}_B)\, |\bar{k}_A \bar{k}_B\rangle$ describes the situation



(for appropriate $\bar{p}_A$, $\bar{p}_B$ and such that the A/B regions overlap at $t=0$)

- More realistically, we need to describe a situation with incoming particles spread over a certain transverse region (i.e. with impact parameters $\bar{b}$):



$(b_3 = 0)$

- As in QM, the operator $\hat{\bar{P}}$ (cf. Noether theorem) generates shifts. Thus, the incoming particles with non-zero $\bar{b}$ are given by:

$$e^{-i\hat{\bar{P}}\bar{b}} \int d\tilde{k}\, f_{\bar{p}_B}(\bar{k}_B)|\bar{k}_B\rangle$$

- Since $\hat{\bar{P}}|\bar{k}_B\rangle = \bar{k}_B|\bar{k}_B\rangle$, an initial state with impact parameter $\bar{b}$ reads

$$|i_b\rangle = \int d\tilde{k}_A d\tilde{k}_B\, f_{\bar{p}_A}(\bar{k}_A) f_{\bar{p}_B}(\bar{k}_B) e^{-i\bar{k}_B\bar{b}} |k_A k_B\rangle$$

- We start with this state "at $t=-\infty$" (when A & B are far apart), evolve forward in time (A & B meet "up to $\bar{b}$" at $t \simeq 0$), and project on the desired scattered state "at $t = +\infty$". The probability for this scattered state (say with momenta p_1, p_2) is

$$|\langle p_1 p_2|S|i_b\rangle|^2.$$

- Summing over a set of particles with different $\bar{b}$ gives

$$N_{events} = \sum_{\bar{b}} |\langle p_1 p_2|S|i_b\rangle|^2.$$

- For a homogeneous transverse distribution of N_B particles in an area F we can approximate the sum by an integral:

$$N_{events} = \frac{N_B}{F} \int d^2b \, |\langle p_1 p_2 | S | i_b \rangle|^2.$$

- The corresponding cross section is

$$\sigma(p_1, p_2) = \frac{N_{events}}{(N_B/F)} = \int d^2b \, |\langle p_1 p_2 | S | i_b \rangle|^2.$$

- Clearly this is too naive since we can not ask for specific final-state momenta p_1, p_2. (This also clashes with the finite precision of any detector.) Instead, we define σ for a finite region of phase space $\mathbb{R}^6 \ni (\bar{p}_1, \bar{p}_2)$: $\sigma(V_f) = \int_{V_f} d\tilde{p}_1 d\tilde{p}_2 \int d^2b \, |\langle p_1 p_2 | \cdots \rangle|^2.$

- The correctness of the proposed measure $d\tilde{p}_1 d\tilde{p}_2$ follows immediately from considering the free theory ($S = \mathbb{1}$) and integrating over the whole phase space:

$$1 \overset{!}{=} \int d\tilde{p}_1 d\tilde{p}_2 \, |\langle p_1 p_2 | f_{\bar{P}_A} f_{\bar{P}_B} \rangle|^2 = \int d\tilde{p}_1 d\tilde{p}_2 \, |f_{\bar{P}_A}(\bar{p}_1)|^2 |f_{\bar{P}_B}(\bar{p}_2)|^2$$

(assuming for simplicity distinguishable particles A & B) ✓

- All of this generalizes to n final state particles. Also, theorists generally leave the actual integration open for as long as possible and talk about <u>differential cross sections ($d\sigma/d\ldots$)</u>:

$$d\sigma = \prod_{j=1}^{n} d\tilde{p}_j \int d^2b \, |\langle p_1 \cdots p_n | S | i_b \rangle|^2.$$

- Now plug in $|i_b\rangle$ as given above and use

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i)\mathcal{M}_{fi}.$$

Unless $f \equiv i$, only the $\mathcal{M}_{fi}$-part contributes. Thus:

$$d\sigma = \prod_j d\tilde{p}_j \int d^2 b \int d\tilde{k}_A d\tilde{k}_B\, f_{\bar{p}_A}(\bar{k}_A) f_{\bar{p}_B}(\bar{k}_B) \int d\tilde{k}_A' d\tilde{k}_B'\, f^*_{\bar{p}_A}(\bar{k}_A') f^*_{\bar{p}_B}(\bar{k}_B')$$
$$e^{i\bar{b}(\bar{k}_B' - \bar{k}_B)} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(p_f - k_i)(2\pi)^4 \delta^4(p_f - k_i')$$

where $p_f = \sum_j p_j$; $k_i = k_A + k_B$; $k_i' = k_A' + k_B'$

- The rest is a fairly straightforward <u>calculation</u>:

$\rightarrow \int d^2 b\, e^{i\bar{b}(\bar{k}_A' - \bar{k}_B)} = (2\pi)^2 \delta^2(k_{B\perp}' - k_{B\perp})$ "$\perp$" $\hat{=}$ transverse (perpendicular)

$\rightarrow \int d^3k_A' d^3k_B'\, \delta^4(p_f - k_i')\, \delta^2(k_{B\perp}' - k_{B\perp}) \cdots$

$$= \int d(k_A^{3\prime})\, d(k_B^{3\prime})\, \delta(p_f^3 - k_i^{3\prime})\, \delta(p_f^0 - k_i^{0\prime}) \cdots$$

- obviously, here $k_{B\perp}' = k_{B\perp}$ from now on
- less obviously, also $k_{A\perp}' = k_{A\perp}$ from now on

(This follows from $\delta^2(p_{f\perp} - k_{A\perp}' - k_{B\perp}')$ together with $\delta^2(p_{f\perp} - k_{A\perp} - k_{B\perp})$.)

$$= \int d(k_A^{3\prime})\, \delta(p_f^0 - k_A^{0\prime} - k_B^{0\prime}) \cdots$$

- obviously, now also $k_B^{3\prime} = p_f^3 - k_A^{3\prime}$

$$= \int d(k_A^{3\prime})\, \delta\left(p_f^0 - \sqrt{m_A^2 + \bar{k}_A'^2} - \sqrt{m_B^2 + \bar{k}_B'^2}\right) \cdots$$

with $\bar{k}_A^{\prime 2} = k_{A\perp}^2 + (k_A^{3\prime})^2$; $\bar{k}_B^{\prime 2} = k_{B\perp}^2 + (p_f - k_A^{3\prime})^2$.

$$\rightarrow \; = \frac{1}{\left| \frac{k_A^{3\prime}}{\sqrt{m_A^2 + \bar{k}_A^{\prime 2}}} - \frac{k_B^{3\prime}}{\sqrt{m_B^2 + \bar{k}_B^{\prime 2}}} \right|} \cdots = \frac{1}{\left| \frac{k_A^{3\prime}}{k_A^{0\prime}} - \frac{k_B^{3\prime}}{k_B^{0\prime}} \right|} \cdots \simeq \frac{1}{|v_A - v_B|} \cdots$$

here we use $k_{A,B} \approx p_{A,B}$, as approximately implemented by $f_{\bar{p}_A}(k_A)$; $f_{\bar{p}_B}(k_B)$.

<u>Note</u>: We talked about "fixed target" initially, but nothing in our analysis depended on $v_A = 0$, so we may as well keep it general.

- We have now completely carried out the $k'_{A,B}$ integrations, in the process implementing the relations

$$k'_{A\perp} = k_{A\perp} \; ; \; k'_{B\perp} = k_{B\perp} \; ; \; k_B^{3\prime} + k_A^{3\prime} = p_f^3$$

together with

$$k_B^{0\prime} + k_A^{0\prime} = p_f^0 \;, \quad \text{i.e.} \;, \sqrt{(k_B^{3\prime})^2 + k_{B\perp}^2 + m_B^2} + \sqrt{(k_A^{3\prime})^2 + k_{A\perp}^2 + m_A^2} = p_f^0$$

- We can view the last two of these 4 relations as two eqs. for the two variables $k_A^{3\prime}$, $k_B^{3\prime}$, which are hence fixed unambiguously.
- Since these two eqs. also hold for k_A^3, k_B^3 due to the (not yet used) δ-fct. $\delta^4(p_f - k_A - k_B)$, we have:

$$\bar{k}_A' = \bar{k}_A \quad \& \quad \bar{k}_B' = \bar{k}_B .$$

- Thus, summarizing, we have arrived at

$$d\sigma = \prod_j d\tilde{p}_j \int d\tilde{k}_A d\tilde{k}_B \, |f_{\bar{p}_A}(\bar{k}_A)|^2 \, |f_{\bar{p}_B}(\bar{k}_B)|^2 \, |\mathcal{M}_{fi}|^2 \cdot \frac{(2\pi)^4 \delta^4(p_f - k_A - k_B)}{4 k_A^0 k_B^0 \, |v_A - v_B|}$$

- We can now view the two "$|f|^2$" as effective δ-fcts. ensuring $\bar{k}_A = \bar{p}_A$ & $\bar{k}_B = \bar{p}_B$, thus

$$d\sigma = \underbrace{\frac{1}{4 p_A^0 p_B^0 \, |v_A - v_B|}}_{\text{This prefactor is invar. under boosts along } x^3 \text{ and can be written as}} |\mathcal{M}_{fi}|^2 \underbrace{(2\pi)^4 \delta^4(p_f - p_i) \prod_{j=1}^{n} \frac{d^3 p_j}{(2\pi)^3 2 p_j^0}}_{n\text{-particle phase space}}$$

$$\frac{1}{2 w(s, m_A^2, m_B^2)}$$ where $w(x,y,z) \equiv \sqrt{x^2 + y^2 + z^2 - 2xy - 2xz - 2yz}$,

($\rightarrow$ problems). and $s \equiv (p_A + p_B)^2$ (centre-of-mass energy^2)

- For us, at the moment the highly relativistic case will be most relevant. Thus, let $p^0 = m_A$; $\bar{p}_A = 0$

$|v_A - v_B| = |v_B| = c = 1$;

$$4 p_A^0 p_B^0 = 2 (p_A + p_B)^2 = 2s$$

$$\Rightarrow \boxed{d\sigma = \frac{1}{2s}|\mathcal{M}_{fi}|^2 dX^{(n)}}$$ for the highly relativistic case

(where $dX^{(n)} = (2\pi)^4\delta^4(..)\, d\tilde{p}_1 \cdots d\tilde{p}_n$)

5

11.3 2-particle phase space & simple example

- Look at "$A+B \to 1+2$" in $\lambda\varphi^4$-theory
- Focus just on phase space first:

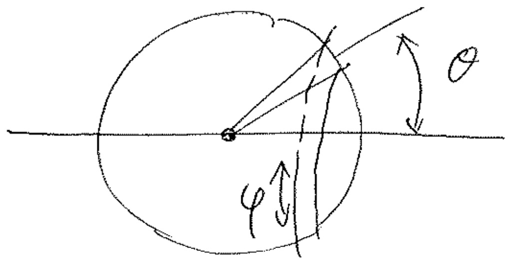
$$\int dX^{(2)} = \int (2\pi)^4\delta^4(p_1+p_2-p_A-p_B)\frac{d^3p_1}{(2\pi)^3 2p_1^0}\cdot\frac{d^3p_2}{(2\pi)^3 2p_2^0}$$

carry out $\int d^3p_1$ trivially and use $\bar{p}_1 = -\bar{p}_2$ (cms-frame)

$$\Rightarrow \int \frac{d^3p_2}{(2\pi)^2 4p_1^0p_2^0}\delta(p_1^0+p_2^0-\sqrt{s}) = \int\frac{d^3p_2}{(2\pi)^2 4|\bar{p}_2|^2}\delta(2|\bar{p}_2|-\sqrt{s})$$

$$d^3p_2 = d\Omega\, |\bar{p}_2|^2 d|\bar{p}_2|$$

$$d\Omega = d\varphi \sin\theta\, d\theta$$



$$\Rightarrow \int d|\bar{p}_2|\,\delta(2|\bar{p}_2|-\sqrt{s})\cdot\frac{d\Omega}{16\pi^2} = \frac{d\Omega}{32\pi^2}$$ (Interpreting "$\int$" without angular integration.)

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s} = \frac{\lambda^2}{64\pi^2 s}$$

(Note: $\frac{\lambda^2}{s^2}$ could have been argued without <u>any</u> calculation!)