6 LSZ-formalism

(Lehmann, Symanzik, Zimmermann; Nuovo Cimento, 1 ('55) p. 205)

- My presentation will follow Peskin/Schröder (which is similar to Weinberg I), but differ from many other books (like Itzykson/Zuber) and my old script.

- General Idea: S-matrix-elements $\longleftrightarrow$ Greens-fcts.

$$_{out}\langle p_1' \cdots p_n' | p_1 \cdots p_m \rangle_{in} \longleftrightarrow \langle 0 | T \varphi(x_1) \cdots \varphi(x_{n+m}) | 0 \rangle$$

needed for cross sect. — easily calculable in pert. th. (e.g. from Feynman rules)

6.1 Spectral density & Z-factor

- abandon Interaction Picture & switch to Heisenberg Picture (even for the interacting theory). Thus, the time dependence of φ will include the full quantum dynamics: $\varphi(x) = e^{iHt} \varphi_s(\bar{x}) e^{-iHt}$; $H = H_0 + H_{int}$

  $|\psi\rangle = |\psi_s(t=0)\rangle$ - time independent

- Consider the "correlation fct." $\langle 0 | \varphi(x) \varphi(y) | 0 \rangle$.

  [We already encountered the free-field version of this in our discussion of causality. For $x^0 > y^0$, $(x-y)^2 > 0$, we can interpret this as amplitude for particle propagation from $\bar{y}$ at time y^0 to $\bar{x}$ at time x^0.]

- For a free field φ_0 with mass m_0 we have

$$\langle 0|\varphi_0(x)\varphi_0(y)\rangle = \int d\tilde{p}\, d\tilde{q}\, \langle 0|a_{\bar{p}}\, a^+_{\bar{q}}|0\rangle\, e^{-ipx+iqy}$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^3} e^{-ip(x-y)}\, \delta(p^2-m_0^2)\, \Theta(p^0)$$

- Now the general case: $\equiv \underline{\underline{D(x-y, m_0^2)}}$

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \sum_\alpha \langle 0|\varphi(x)|\alpha\rangle\langle\alpha|\varphi(y)|0\rangle$$

↑ all states, incl. multi-particle

(The sum is symbolic, it includes integration over the continuous indices present e.g. because of boosts & rotations.)

- Use $\varphi(x) = e^{i\hat{P}x}\varphi(0)e^{-i\hat{P}x}$; $e^{-i\hat{P}x}|\alpha\rangle = e^{-ip_\alpha x}|\alpha\rangle$

$$\Rightarrow \cdots = \sum_\alpha e^{-ip_\alpha(x-y)}\, |\langle 0|\varphi(0)|\alpha\rangle|^2$$

$$= \int d^4q \sum_\alpha e^{-iq(x-y)}\, |\langle 0|\varphi(0)|\alpha\rangle|^2\, \delta^4(q-p_\alpha)$$

$$= \int \frac{d^4q}{(2\pi)^3}\, e^{-iq(x-y)}\, \rho(q)$$

where $\rho(q) \equiv (2\pi)^3 \sum_\alpha \delta^4(q-p_\alpha)\, |\langle 0|\varphi(0)|\alpha\rangle|^2$

- $\rho(q)$ is manifestly $SO^+(1,3)$-inv. & zero for $q^0<0$. Hence:

$$\rho(q) \equiv \Theta(q^0)\, \underline{\sigma(q^2)}$$

This "<u>spectral density</u>" quantifies the contribution of intermediate states $|\alpha\rangle$ with $p_\alpha^2 = q^2$ to the correlation fct.

- Further rewriting gives:

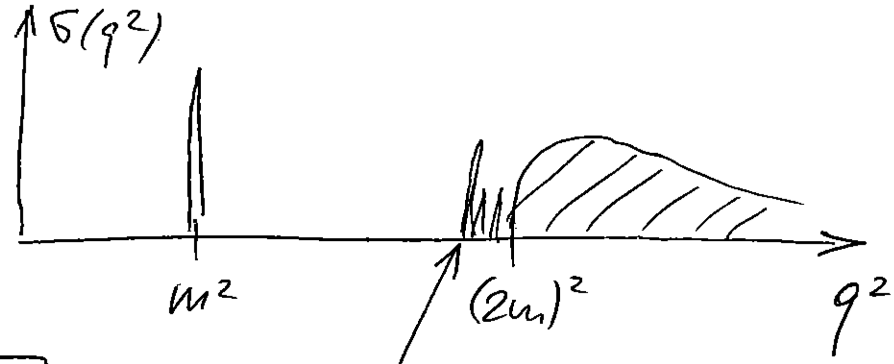
$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int_0^\infty dM^2 \int \frac{d^4q}{(2\pi)^3}\, e^{-iq(x-y)}\, \delta(q^2-M^2)\,\theta(q^0)\,\sigma(M^2)$$

$$= \int_0^\infty dM^2\, D(x-y, M^2)\,\sigma(M^2)$$

(cf. the formula derived above for the free corr. fct.)

- We see that $\sigma(q^2) = \delta(q^2 - m_0^2)$ for the free theory.

More generally:



multi-particle bound states or "resonances", starting at the "multi-particle threshold" M_t^2.

Note: We assume that the vacuum does not contribute as an intermediate state, $\langle 0|\varphi(x)|0\rangle = 0$. This can be ensured by redefinition $\varphi \to \varphi + \text{const.}$

- Thus: $\sigma(q^2) = \underset{\text{normaliz. factor}}{Z}\, \delta(q^2 - \underset{\neq m_0^2 \text{ due to interactions}}{m^2}) + \underbrace{\cdots}_{\text{non-zero for } q^2 > M_t^2}$

- To be more precise about Z:

$$\sum_\alpha |\alpha\rangle\langle\alpha| = \underbrace{\int d\tilde{p}\, |p\rangle\langle p|}_{\text{single-particle states of interacting theory}} + \cdots$$

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int d\tilde{p}\, \langle 0|\varphi(x)|p\rangle\langle p|\varphi(y)|0\rangle + \cdots$$

$$= \int d\tilde{p}\, e^{-ip(x-y)}\, \underbrace{|\langle 0|\varphi(0)|p\rangle|^2}_{\equiv Z} + \cdots$$

$$= D(x-y, m^2) Z + \cdots$$

- Thus, finally:

$$\boxed{\langle 0|\varphi(x)\varphi(y)|0\rangle = Z D(x-y, m^2) + \int_{M_t^2}^{\infty} dM^2 \sigma(M^2) D(x-y, M^2)}$$

- Subtract the same eq. with $x \leftrightarrow y$ and use the definition

$\Delta(x-y, m_0^2) \equiv \langle 0|[\varphi_0(x), \varphi_0(y)]|0\rangle$. Obtain:

$$\langle 0|[\varphi(x), \varphi(y)]|0\rangle = Z \Delta(x-y, m^2) + \int_{M_t^2}^{\infty} dM^2 \sigma(M^2) \Delta(x-y, M^2)$$

- Since $\dot{\varphi} = \pi$, applying $\frac{\partial}{\partial y^0}\Big|_{y^0 = x^0}$ to both sides of this eq., we get

$$[\varphi(x^0, \bar{x}), \pi(x^0, \bar{y})] = i\delta^3(\bar{x}-\bar{y}),$$

and the same on the r.h. side (for the free theory). Hence

$$1 = Z + \int_{M_t^2}^{\infty} dM^2 \sigma(M^2).$$

$\Rightarrow Z \leq 1$ and $Z < 1$ if theory is not free. The size of $1-Z$ accounts for the overlap of $\varphi(0)|0\rangle$ with multi-particle states.

- Finally, in analogy to our transition $\varphi(x)\varphi(y) \to [\varphi(x), \varphi(y)]$, we can also consider $T\varphi(x)\varphi(y)$:

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = Z D_F(x-y, m^2) + \int_{M_t^2}^{\infty} dM^2 \sigma(M^2) D_F(x-y, M^2).$$

The "Feynman propagator" $D_F(x-y, m_0^2) \equiv \langle 0|T\varphi_0(x)\varphi_0(y)|0\rangle$ will be especially important below.

6.2 LSZ reduction formula

- Now we will relate time-ordered correl. fcts. to scattering amplitudes. We start by considering the Fourier trf.

$$\int d^4x\, e^{ipx} \langle 0| T\varphi(x)\varphi(z_1)\cdots\varphi(z_n)|0\rangle$$

$$= \int_{-\infty}^{T_-} dx^0 (\cdots) + \int_{T_-}^{T_+} dx^0 (\cdots) + \int_{T_+}^{\infty} dx^0 (\cdots). \qquad \boxed{(T_- < z_i^0 < T_+)\ \forall i}$$

- We will only need the pole structure in p_0 of this expression (viewed as a fct. of the complex variable p_0). In particular, we claim that there is a pole at $p_0 = \sqrt{\bar{p}^2+m^2}$ and we determine its residue. Focus first on the last int. region:

$$\int_{T_+}^{\infty} dx^0 (\cdots) = \int_{T_+}^{\infty} dx^0 \int d^3x\, e^{ipx} \sum_\alpha \langle 0|\varphi(x)|\alpha\rangle\langle\alpha| T\varphi(z_1)\cdots\varphi(z_n)|0\rangle$$

$$= \sum_\alpha \int_{T_+}^{\infty} dx^0 \int d^3x\, e^{ix^0(p^0-q^0_\alpha) - i\bar{x}(\bar{p}-\bar{q}_\alpha)} \langle 0|\varphi(0)|\alpha\rangle\langle\cdots\rangle.$$

- A pole in p^0 arises if the oscillating exponent vanishes, i.e. if $p^0 - q^0_\alpha = 0$. Thus, a pole at $p_0 = \omega_{\bar{p}}$ can only come from 1-particle-states ($\int d^3x$ forces $\bar{q}_\alpha = \bar{p}$ and only 1-part.-states then have $q^0_\alpha = \omega_{\bar{p}}$.).

- Thus, we can replace $\sum_\alpha |\alpha\rangle\langle\alpha| \longrightarrow \int d\tilde{q}_\alpha\, |q_\alpha\rangle\langle q_\alpha|$

$$\Rightarrow \;\ldots \sim \int_{T_+}^{\infty} dx^0 \int d^3x \int d\tilde{q}\, e^{ix^0(p^0-q^0_\alpha) - i\bar{x}(\bar{p}-\bar{q})} \langle 0|\varphi|q\rangle\langle q|\cdots\rangle$$

"up to finite terms"

$$\ldots \sim \frac{\sqrt{Z}}{2\omega_{\bar{p}}} \int_{T_+}^{\infty} dx^0\, e^{ix^0(p^0-\omega_{\bar{p}})} \langle p|\cdots\rangle$$

<u>Note:</u> Allow for small pos. imag. part of p^0 to make integral well-defined at $x^0 \to \infty$, then analyt. continue to desired value of p^0 ...

$$\ldots \sim \frac{\sqrt{Z}}{2\omega_{\vec{p}}} \cdot \frac{1}{i(p^0-\omega_{\vec{p}})} \left(0 - e^{ix^0(p^0-\omega_{\vec{p}})}\Big|_{T_+} \right) \langle p | \cdots \rangle$$

$$\boxed{\sim \frac{i\sqrt{Z}}{p^2-m^2} \langle p | T \varphi(z_1) \cdots \varphi(z_n) | 0 \rangle}$$

pole-structure at $p^0 \sim \omega_{\vec{p}}$.

- First integr. region: $\int_{-\infty}^{T_-} dx^0 \ldots \Rightarrow \langle \cdots \rangle \langle \alpha | \varphi(x) | 0 \rangle$

  $\Rightarrow e^{ix^0(p^0+\omega_{\vec{p}})} \Rightarrow$ pole at $p^0 = -\omega_{\vec{p}}$, not relevant

  (<u>Note:</u> the above formula does in fact continue to hold, but we here only care about $p^0 \sim \omega_{\vec{p}}$.)
- Middle integr. region: finite, hence analytic in p^0, no poles.

<u>Prelim. result:</u> We managed to trade $\varphi(x)$ for outgoing particle $\langle p | \cdots$.

<u>Completely analogously:</u>

$$\int d^4x \, e^{-ikx} \langle 0 | T \varphi(x) \varphi(z_1) \cdots \varphi(z_n) | 0 \rangle$$

different sign in exponent!

$$\sim \frac{i\sqrt{Z}}{k^2-m^2} \langle 0 | T \varphi(z_1) \cdots \varphi(z_n) | k \rangle$$

pole structure at $k^0 \sim \omega_{\vec{k}}$.

- Finally, we need to be able to do *several* such manipulations at once, e.g. to derive

$$\int_{x_1}\int_{x_2} e^{ip_1x_1+ip_2x_2} \langle 0|T\varphi(x_1)\varphi(x_2)\varphi(z_1)\cdots\varphi(z_n)|0\rangle$$

$$\sim \frac{i\sqrt{Z}}{p_1^2-m^2}\cdot\frac{i\sqrt{Z}}{p_2^2-m^2}\langle p_1p_2|T\varphi(z_1)\cdots|0\rangle .$$

- The derivation would go through as above if we had

$$\langle 0|T\varphi(x_1)\varphi(x_2)|p_1p_2\rangle = \langle 0|\varphi(x_1)|p_1\rangle\langle 0|\varphi(x_2)|p_2\rangle$$

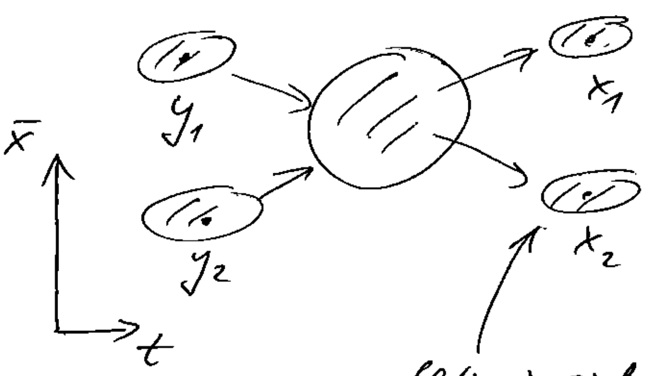
etc.

- This will be true if $\bar{x}_1, \bar{x}_2$ are always far apart at $x_1^0, x_2^0 \to +\infty$. For a double Fourier-trf. in x_1, x_2 this is certainly not true. Hence, we need to use wave packets:

$$\int d^4x\, e^{ipx} \longrightarrow \int d^4x \underbrace{\int d\tilde{k}\, f_{\bar{p}}(\tilde{k})e^{ikx}}$$

fct. of $\bar{x}$ that is localized near zero at $x^0 \simeq 0$ (and correspondingly at other values of x^0).

Vis.:



$\varphi(x_2)$ only contributes for x_2 in this region & analogously for other fields

$\Rightarrow$ Fold $\langle 0|T\varphi(z_1)\cdots\varphi(z_n)|0\rangle$ with approp. wave packets, show that result is approp. combination of poles with the residue being the desired matrix element. Take limit $f_{\bar{p}}(\bar{k}) \to \delta^3(\bar{k}-\bar{p})$ at the end.

(Note: Doing this last limit rigorously is non-trivial, but if it works, the result is clear ...)

LSZ-reduction formula:

$$\prod_{i=1}^{n} \int d^4x\, e^{ip_i x_i} \prod_{j=1}^{m} \int d^4y\, e^{-ik_j y_j} \langle 0|T\, \varphi(x_1)\cdots\varphi(x_n)\varphi(y_1)\cdots\varphi(y_m)|0\rangle$$

$$\underset{\substack{p_i^0 \to \omega_{\bar{p}_i} \\ k_i^0 \to \omega_{\bar{k}_i}}}{\sim} \prod_{i=1}^{n} \frac{i\sqrt{Z}}{p_i^2-m^2} \cdot \frac{i\sqrt{Z}}{k_i^2-m^2} \underbrace{{}_{out}\langle p_1 \cdots p_n | k_1 \cdots k_m \rangle_{in}}_{\equiv S_{fi}},$$

overriding our previous definition.

Crucial: $|k_1 k_2\rangle_{in}$ means a state in the Heisenberg picture of the fully interacting theory which physically corresponds to two incoming wave-packets of particles $|k_1\rangle, |k_2\rangle$ (separated in $\bar{x}$) at $x^0 \to -\infty$. (Analogously for ${}_{out}\langle \cdots |$.

We will calculate S_{fi}'s by Fourier-transforming time-ordered correl. fcts. and extracting the relevant residue.

6.3 Calculating time-ordered correl. fcts.

- We need $\langle 0|T\varphi(x_1)\cdots\varphi(x_n)|0\rangle$ for any n.

- For simplicity, start with $n=2$ and call $x_1 = x'$; $x_2 = x$.
(Our argument will go through identically for $n>2$.)

$$\langle 0|T\varphi(x')\varphi(x)|0\rangle = \langle 0|\varphi(x')\varphi(x)|0\rangle \text{ if, W.L.O.G., } t'>t.$$

- Let us suppress the $\bar{x},\bar{x}'$-dependence for notational simplicity & write:

$$\cdots = \langle 0|\varphi(t')\varphi(t)|0\rangle = \langle 0|e^{iHt'}\varphi(0)e^{-iH(t'-t)}\varphi(0)e^{-iHt}|0\rangle$$

$$= \langle 0|\underbrace{e^{iHt'}e^{-iH_0t'}}_{U(0,t')}\underbrace{e^{iH_0t'}\varphi(0)e^{-iH_0t'}}_{\varphi_I(t')}\underbrace{e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}}_{U(t',t)}\underbrace{e^{iH_0t}\varphi(0)e^{-iH_0t}}_{\varphi_I(t)}\underbrace{e^{-iHt}}_{U(t,0)}|0\rangle$$

This is the already familiar unitary operator evolving states in the interaction picture.

$$= \langle 0|U(0,t')\varphi_I(t')U(t',t)\varphi_I(t)U(t,0)|0\rangle$$

$$\boxed{\begin{array}{l}\text{more correctly:}\\ U(-T,\infty)\ \&\ T\to\infty\end{array}}$$

$$= \langle 0|U(0,\infty)U(\infty,t')\varphi_I(t')U(t',t)\varphi_I(t)U(t,-\infty)U(-\infty,0)|0\rangle$$

- Let us now assume that the interaction is adiabatically switched off at $t\to\pm\infty$ ($H_{int}(t)\to f(t)H_{int}(t)$ with appropriately chosen fct. f). QM-evolution is still unitary and, by adiabaticity, the true (interacting) vacuum evolves to the $\underbrace{\text{free vacuum}}$ in this limit.

$|0\rangle_I \equiv$ state annihilated by all $a_{\bar{p}}$, related in familiar way to φ_I.

$$\Rightarrow \mathcal{U}(t_1,-\infty)\mathcal{U}(-\infty,0)|0\rangle = \mathcal{U}(t_1,-\infty)|0\rangle_I \, {}_I\langle 0|\mathcal{U}(-\infty,0)|0\rangle$$

$$\Rightarrow \langle 0|T\varphi(t')\varphi(t)|0\rangle = \frac{{}_I\langle 0|\mathcal{U}(\infty,t')\varphi_I(t')\mathcal{U}(t',t)\varphi_I(t)\mathcal{U}(t,-\infty)|0\rangle_I}{\left(\langle 0|\mathcal{U}(0,\infty)|0\rangle_I \, {}_I\langle 0|\mathcal{U}(-\infty,0)|0\rangle\right)^{-1}}$$

- The denominator is a product of two phases (due to unitarity of $\mathcal{U}$ & adiabaticity), hence $(\cdot\cdot)^{-1} = (\cdot\cdot)^*$. Thus, the denominator is:

$$\langle 0|\mathcal{U}(0,\infty)|0\rangle_I^* \, {}_I\langle 0|\mathcal{U}(-\infty,0)|0\rangle^* = \langle 0_I|\mathcal{U}(\infty,0)|0\rangle\langle 0|\mathcal{U}(0,-\infty)|0\rangle_I$$

$$= \langle 0_I|\mathcal{U}(\infty,-\infty)|0\rangle_I$$

- Now recall our previous result

$$\mathcal{U}(t',t) = T\exp\left[-i\int_t^{t'} d\tau\, H_{int}(\varphi_I(\tau))\right].$$

Insert this in both numerator & denominator. Note that, in the denominator, everything is time-ordered automatically such that we can drop the separate T's and instead put one single T at the front:

$$\underbrace{{}_I\langle 0|T\exp\left(-i\int_{t'}^{\infty} H_{int}d\tau\right)\varphi_I(t')\exp\left(-i\int_t^{t} H_{int}d\tau\right)\varphi_I(t)\cdots|0\rangle_I}_{\text{under } T\text{, the order doesn't matter, hence can write product of } \varphi\text{'s \& one single exponent.}}$$

- Collecting everything and generalizing to many fields:

$$\boxed{\langle 0|T\varphi(x_1)\varphi(x_2)\cdots|0\rangle = \frac{{}_I\langle 0|T\varphi_I(x_1)\varphi_I(x_2)\cdots\exp\left[-i\int_{-\infty}^{\infty}d\tau\, H_{int}(\varphi_I)\right]|0\rangle_I}{{}_I\langle 0|T\exp\left[-i\int_{-\infty}^{\infty}d\tau\, H_{int}[\varphi_I]\right]|0\rangle_I}}$$

Note: Can take $f(t)\to 1$ in this final result

This is HUGE progress since, on the r.h. side, everything can be evaluated in terms of commutators of free fields.