8 Electromagnetic field

8.1 Gauge invariance

- The complex scalar, with $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi\phi^*$, has a (global) $U(1)$-symm.: $\phi(x) \to e^{i\alpha}\phi(x)$.
- "global" means that the trf. is the same "for the whole world".
- Following our "locality paradigm", we would like to promote this to a "local" or "gauge" symmetry:
$$\phi(x) \to e^{i\alpha(x)}\phi(x).$$
- The derivative of ϕ now transforms as
$$\partial_\mu \phi \to \partial_\mu (e^{i\alpha}\phi) = e^{i\alpha}\left(\partial_\mu\phi + i(\partial_\mu \alpha)\phi\right).$$
- This is not equal to $e^{i\alpha}\partial_\mu\phi$ such that, unlike the global case, the phase does not simply drop out and $\mathcal{L}$ is not invariant.
- To fix this, one introduces a "gauge connection" $A_\mu(x)$ and defines the "covariant derivative" $\boxed{D_\mu \equiv \partial_\mu + iA_\mu(x)}$
- The covariant derivative of ϕ transforms as
$$\begin{aligned} D_\mu\phi \to D'_\mu\phi' &= (\partial_\mu + iA'_\mu)e^{i\alpha}\phi \\ &= e^{i\alpha}\left(\partial_\mu\phi + i(\partial_\mu\alpha)\phi + iA'_\mu\right)\phi \\ &\overset{!}{=} e^{i\alpha}D_\mu\phi = e^{i\alpha}(\partial_\mu\phi + iA_\mu\phi) \end{aligned}$$

$\Rightarrow$ We must demand: $\boxed{A'_\mu = A_\mu - \partial_\mu\alpha}$

- Now $D_\mu \phi$ transforms with a phase, just like ϕ, and $\mathcal{L}$ is invariant.

Comment: A conventional derivative is defined as

$$n^\mu \partial_\mu \phi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\phi(x+\epsilon n) - \phi(x)\right).$$

But in the presence of local gauge symm. this makes no sense since the phases of $\phi(x)$ & $\phi(x+\epsilon n)$ are independent and so we can not "compare" these two quantities. If we have a way to "parallel transport ϕ:

$$\phi(x) \longrightarrow \overbrace{U(y,x)}^{\in U(1)}\phi(x),$$

such that $U(y,x)' = e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$ and hence $U(y,x)\phi(x)$ transforms like a field at y, we can compare:

$$n^\mu D_\mu \phi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\phi(x+\epsilon n) - U(x+\epsilon n, x)\phi(x)\right).$$

If we assume U to be smooth & $U(x,x) = 1$, we have $U(x+\epsilon n, x) \underset{\uparrow}{=} 1 - i\epsilon n^\mu A_\mu(x) + \cdots$

here we define A_μ by the lin. coeff. of the Taylor exp. of U.

The resulting D_μ and the tr. properties of A_μ agree with our starting definition.

Furthermore, given some $A_\mu(x)$, we can define $U(y,x) = \exp\left(i\int_x^y A_\mu dz^\mu\right)$ with (e.g., but not necessarily) the straight line connecting x & y. This is called a "Wilson line". The name gauge connection ("connecting points") should now be more clear.

- Having introduced A_μ, we also need to specify its dynamics, i.e., a gauge-inv. action for A_μ.
- To do so, observe that the diff. operator D_μ transforms as

$$D_\mu \xrightarrow{\alpha} D_\mu' = e^{i\alpha} D_\mu e^{-i\alpha}$$

$$\left(\text{Check: } e^{i\alpha}(\partial_\mu + iA_\mu)e^{-i\alpha} = e^{i\alpha}(\partial_\mu e^{-i\alpha}) + \partial_\mu + iA_\mu \right.$$

$$\left. = \partial_\mu + iA_\mu' \checkmark \right).$$

- Hence $[D_\mu, D_\nu] \rightarrow e^{i\alpha}[D_\mu, D_\nu]e^{-i\alpha}$.
- But, at the same time

$$[D_\mu, D_\nu] = \partial_\mu\partial_\nu + \partial_\mu iA_\nu + iA_\mu\partial_\nu - A_\mu A_\nu - \{\mu \leftrightarrow \nu\}$$

$$= i(\partial_\mu A_\nu) + iA_\nu\partial_\mu + iA_\mu\partial_\nu - \{\mu \leftrightarrow \nu\}$$

$$= iF_{\mu\nu} \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Thus, contrary to its appearance, $[D_\mu, D_\nu]$ is <u>not</u> a diff. operator (derivatives acting right have dropped out). $\Rightarrow$ $[D_\mu, D_\nu] \to e^{i\alpha}[D_\mu, D_\nu]e^{-i\alpha}$ implies

$$F_{\mu\nu} \to e^{i\alpha}F_{\mu\nu}e^{-i\alpha} = F_{\mu\nu}$$

- $F_{\mu\nu}$ is gauge-inv. & our proposed <u>Scalar QED</u> Lagrangian is

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - m^2|\phi|^2,$$

which is inv. under $\phi \to e^{i\alpha(x)}\phi$; $A_\mu \to A_\mu - \partial_\mu\alpha$.
It is also Poinc. inv. if we define

Shift d^μ: $\quad A_\mu'(x) = A_\mu(x-d)$

Lorentz-tf. Λ: $\quad A_\mu'(x) = \underbrace{\Lambda_\mu{}^\nu A_\nu(\Lambda^{-1}x)}$

We saw earlier that the vector field $\partial_\mu\varphi$ constructed from φ transforms in this way. Here we declare A_μ to be a fundamental vector field having this property by definition.

- A possibly more familiar form of $\mathcal{L}$ arises after the field redefinition $A_\mu \to e A_\mu$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D\phi|^2 - m^2 |\phi|^2 \; ; \; D_\mu = \partial_\mu + ie A_\mu .$$

(In this form it is more apparent that e is a coupling constant.)

<u>Advanced comment:</u> Using the diff. form language, A is a 1-form ($A = A_\mu dx^\mu$); α a 0-form; the gauge trf. is $A \to A + d\alpha$; $F = dA$ is a 2-form; the action is $\mathcal{L} \sim F \wedge * F$ with "$*$" the <u>Hodge-operator</u>

8.2 Gupta-Bleuler-Quantization

We focus just on the free part of the action, finding

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \; ; \quad \pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \frac{\partial}{\partial(\partial_0 A_\mu)} \left(-\frac{1}{4} F_{\rho\sigma} F_{\sigma\tau} \eta^{\rho\sigma} \eta^{\nu\tau} \right)$$

Quantize like system of four indep. fields (e.g. φ_i, $i = 1..4$)

Check yourself! *)

$$\underline{\underline{\pi^\mu = F^{\mu 0}}}$$

*) Use $\dfrac{\partial}{\partial(\partial_0 A_\mu)} (\partial_\rho A_\nu) = \eta^0{}_\rho \eta^\mu{}_\nu$

- In particular, we find $\underline{\pi^0 = 0}$, which is a <u>problem</u>.

(For a deeper understanding of this & systematic ways to proceed, one needs to understand the <u>quantization of systems w/ constraints</u>. See e.g. Dirac's famous "Lectures on QM", '64. See also Kugo: "Eichtheorie" & many others.)

- We proceed "naively" by fixing the gauge:
  Lorentz or covariant gauge: Demand $\underline{\partial A \equiv \partial_\mu A^\mu = 0}$.

- Use new Lagrangian $\mathcal{L} = -\frac{1}{4}F^2 - \frac{\lambda}{2}(\partial A)^2$.
  (This is ok since the old & new EOM's are the same if $\partial A = 0$.)

- Choosing $\lambda = 1$ makes $\mathcal{L}$ particularly simple:

$$\mathcal{L} = -\frac{1}{4}\left(2(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \underbrace{2(\partial_\mu A_\nu)(\partial^\nu A^\mu)}_{\text{integrate by parts \& drop total derivatives}}\right) - \frac{1}{2}(\partial A)^2$$

$$= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial A)^2 - \frac{1}{2}(\partial A)^2$$

$$= \underbrace{\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A_\rho)(-\eta^{\nu\rho})}$$

precisely like 4 real scalars A_ν, but with wrong overall sign for A_0.

- By explicit calculation (as above) or by analogy to real scalar: $\pi^\mu = -\dot{A}^\mu = \underbrace{(-\eta^{\mu\nu})}_{\text{wrong sign for } A_0}\dot{A}_\nu$

- We quantize by demanding

$$[A,A] = [\pi,\pi] = 0 \quad \& \quad [A_\mu(\bar{x}), \pi^\nu(\bar{y})] = i\eta_\mu{}^\nu \delta^3(\bar{x}-\bar{y})$$

$$(\eta_\mu{}^\nu = \delta_\mu{}^\nu)$$

- Fourier transforming; introducing a, a^+; determining their commut. relations; going over to Heisenberg-fields works as before. The result is (Check if not certain!):

$$A_\mu(x) = \int d\tilde{k}\left(a_{\bar{k},\mu} e^{-ikx} + a^+_{\bar{k},\mu} e^{ikx}\right) \; ; \quad k^0 = |\bar{k}|$$

$$[a,a] = [a^+,a^+] = 0 \; ; \quad [a_{\bar{k},\mu}, a^+_{\bar{k}',\nu}] = -\eta_{\mu\nu}\, 2k^0 (2\pi)^3 \delta^3(\bar{k}-\bar{k}')$$

↑ Note, again, the wrong sign for the A_0-operators.

- Definitive check: Obtain π^μ from A_μ above and, using a/a^+ commut. relations, determine that π, A really satisfy the canonical commut. rels. we postulated.

- Logical next step: Define $|0\rangle$ as state annihilated by $a_{\bar{k},\mu}$ $(\forall \bar{k},\mu)$ & construct Fock space by applying a^+'s of all four types. But...

<u>Problem ①</u>: For $a_{\bar{k},0}$, the wrong sign causes trouble. To see this without much writing, drop "$\bar{k}$" & consider "harm. osc." algebra with wrong sign: $[a,a^+] = -1$: Focus on 1st exc. states $\|a^+|0\rangle\|^2 = \langle 0|aa^+|0\rangle = \langle 0|(a^+a-1)|0\rangle$

$$= -1$$

Such a non-pos.-def. Hilbert-space metric or scalar product is unacceptable in QM. Also, we can not simply "switch the roles of a_0 & a_0^+" to fix the sign since the <u>relative sign</u> between $[a_0, a_0^+]$ & $[a_i, a_i^+]$ is enforced by Lorentz-symm.

<u>Problem ②</u>: Declaring $\partial A = 0$ at the operator level (which we at least naively should do) is impossible since:

$$[A_0, \partial A] = [A_0, \dot{A}_0] \neq 0 .$$

<u>Solution</u>: (Gupta, Bleuler) [we follow Nakanishi's Book]

- Let F be the Fock-space constructed above & define $F_{phys} \subset F$ by $\partial A^{a} |\psi\rangle = 0 \Leftrightarrow |\psi\rangle \in F_{phys}$ ("physical", a = annih. part)
- This implies $\langle\psi|\partial A|\psi\rangle = \langle\psi|(\partial A^a + \partial A^c)|\psi\rangle$

$$= \langle\psi|(\partial A^a + (\partial A^a)^+|\psi\rangle = 0 \quad \text{if } |\psi\rangle \in F_{phys}$$

vanishes after acting left.

$\Rightarrow$ gauge condition is satisfactorily implemented on F_{phys}.

- It turns out that F_{phys} does not contain neg.-norm-states any more, but it does contain a zero-norm subspace $F_0 \subset F_{phys}$. (F_{phys} is "positive-semi-definite.")

We define: $\underset{\text{Hilbert space}}{\mathcal{H}} \equiv \underbrace{F_{phys} / F_0}_{\text{space of equivalence classes, "}\sim\text{",}}$

where $|\psi\rangle \sim |\psi'\rangle \Leftrightarrow \| |\psi\rangle - |\psi'\rangle \| = 0$.

- Before proceeding, let's make a detour about

Polarizations:

- general 1-photon state: linear combination of the four states $a^+_{\bar{k},\mu}|0\rangle$; $\mu = 0..3$. Write this as

$$-\varepsilon^\mu(k)\, a^+_{\bar{k},\mu}|0\rangle \qquad (\bar{k} \text{ fixed!})$$

- Obviously, the space of $\varepsilon^\mu(k)$ is 4-dimensional and we can choose some basis. It is convenient to demand "covariant orthonormality":

$$\varepsilon^{(\lambda)}_\mu(k)\cdot\left(\varepsilon^{(\lambda')\mu}(k)\right)^* = \eta^{\lambda\lambda'}$$

Problem: Prove that orthonormality implies completeness, i.e.

$$\sum_{\lambda,\lambda'} \eta_{\lambda\lambda'}\, \varepsilon^{(\lambda)}_\mu(k)\left(\varepsilon^{(\lambda')}_\nu(k)\right)^* = \eta_{\mu\nu}$$

- To make a concrete choice, introduce some arbitrary but fixed unit vector with positive time component:

$$n = \{n^\mu\} \text{ with } \underline{n_0 > 0} \;\&\; \underline{n^2 = 1}.$$

- Demand: $\epsilon^{(0)} = n$; $\varepsilon^{(i)}\cdot n = 0$; $\varepsilon^{(1)}\cdot k = \varepsilon^{(2)}\cdot k = 0$.
- To see that this is consistent with our previous requirements on $\varepsilon^{(\lambda)}$, go to a coord. system where $n = (1,\bar{0})$ and $k = (|\bar{k}|, 0, 0, |\bar{k}|)^T$ (recall that $k^2 = m^2 = 0$).
- Now all conditions are met by

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix};\quad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix};\quad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

which is unambiguous up to rotations in the 1-2-plane

- A <u>different</u> common choice of polarization vectors, which still form a basis but do <u>not</u> obey orthonormality is as follows:

- Choose arbitrary but fixed n with $n^2 = 0$ (lightlike); Demand: "unphys." "longit."
$$\varepsilon^u = n \; ; \; \varepsilon^L \sim k \; ; \; \varepsilon^{(1)}, \varepsilon^{(2)} \text{ orthog. to } n \,\&\, k.$$
The previous orthonormality relations are now replaced by
$$(\varepsilon^u)^2 = (\varepsilon^L)^2 = 0 \; ; \quad \varepsilon^u \cdot \varepsilon^L = 1 \; ; \quad \varepsilon^u \cdot \varepsilon^{(i)} = \varepsilon^L \cdot \varepsilon^{(i)} = 0$$
$$\underbrace{\varepsilon^{(i)} \cdot \varepsilon^{(j)} = -\delta^{ij}}_{\text{as before}}$$

- This time we choose a coord. system where $n = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$ and $k = \begin{pmatrix}|\vec{k}|\\0\\0\\|\vec{k}|\end{pmatrix}$. We find:
$$\varepsilon^u = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \; ; \quad \varepsilon^L = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \; ; \quad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \; ; \quad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

- We will write $\varepsilon^\mu(k)\, a^+_{\vec{k},\mu} |0\rangle \equiv |\varepsilon, k\rangle$, which implies:
$$\langle \varepsilon', k' | \varepsilon, k\rangle = \varepsilon'^\mu(k')^* \varepsilon^\nu(k) \langle 0 | a_{\vec{k}',\mu}\, a^+_{\vec{k},\nu} |0\rangle$$
$$= \varepsilon'^\mu(k')^* \varepsilon^\nu(k) \cdot \left(-\eta_{\mu\nu}\, 2k^0 (2\pi)^3 \delta^3(\vec{k}-\vec{k}')\right)$$
$$= -(\varepsilon' \cdot \varepsilon) \cdot 2k^0 (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$
$\Rightarrow$ $(-\varepsilon'\varepsilon)$ measures overlap; $(-\varepsilon^2)$ measures norm of state

- We can now reformulate our "phys.-space condition" $\partial A^a|\psi\rangle = 0$ using polarization vectors:

$$\partial A^a|\varepsilon, q\rangle = 0 \iff k_\mu \cdot a^\mu_{\bar{k}}|\varepsilon \cdot q\rangle = 0 \iff \ldots$$

$$\ldots \iff k_\mu \cdot a^\mu_{\bar{k}} \cdot a^{\nu +}_{\bar{q}} \varepsilon_\nu(q)|0\rangle = 0 \iff \underline{\underline{k \cdot \varepsilon(k) = 0}}$$

- This is violated $\underline{\underline{\text{only}}}$ for ε^u (hence "unphysical").
- We now perform a lin. trf. on the space of our four creation/annih. operators, defining:

$$\alpha^+_{\bar{k}, (u,L,1,2)} \equiv \varepsilon^{(u,L,1,2)\mu}(k) \cdot a^+_{\bar{k},\mu}.$$

We think of F built by applying the four α^+'s to the vacuum. (This is of course still the same F)

- <u>Fact:</u> F_{phys} is the subspace built using only $\alpha^+_{(L,1,2)}$.

(This is clear since the phys. space condition

$$k \cdot a_{\bar{k}} \left(\text{product of various } \varepsilon_{\bar{k}} \cdot a^+_{\bar{k}}\right)|0\rangle = 0$$

can only be violated if an $\varepsilon^u_{\bar{k}}$ appears.)

- Thus $F_{phys.}$ is spanned by states of the type

$$|\psi\rangle = \left(\text{Product of } \alpha^+_{L,1,2}\text{'s}\right)|0\rangle.$$

- $\||\psi\rangle\| = 0$ if and only if <u>at least one α^+_L</u> appears in this product. (In this case also $|\psi\rangle \sim 0$).
- <u>Fact:</u> The presence of such "zero-norm phys. states" does not affect any observable:

$$\langle\psi'|O|\psi'\rangle = \langle\psi|O|\psi\rangle \quad \text{if} \quad |\psi'\rangle = |\psi\rangle + (\cdots \alpha_L^+ \cdots)|0\rangle$$

• This relies on gauge-invariance of observables. We will not give a proof but illustrate this with an important example: $H = :\int d^3x\,(\pi^\mu \dot{A}_\mu - \mathcal{L}): = \dots = \int d\tilde{k}\, k_0\,(-a^+_{\bar{k},\mu}\, a_{\bar{k}}{}^\mu)$

$$= \int d\tilde{k}\, k_0 \Big(\sum_{i=1}^{2} \alpha^+_{\bar{k},i}\alpha_{\bar{k},i} - \underbrace{\left[\alpha^+_{\bar{k},u}\alpha_{\bar{k},L} + \alpha^+_{\bar{k},L}\alpha_{\bar{k},u}\right]}\Big)$$

States with α_L^+-excitations don't contribute to H. $\Leftarrow$ vanishes "inside" any phys. state, $\langle\psi|\cdots|\psi\rangle$.

Summary:

$|\psi\rangle \in F \quad \Rightarrow \quad |\psi\rangle = \sum (\alpha^+_{\bar{k},i}\,\alpha^+_{\bar{q},u}\,\alpha^+_{\bar{p},L}\cdots)|0\rangle$

$|\psi\rangle \in F_{phys.} \quad \Rightarrow$ no terms involving α_u^+ allowed

$|\psi\rangle \in F_0 \quad \Rightarrow$ each term in the sum involves at least one α_L^+-factor (this is obviously a lin. subspace)

$$\mathcal{H} = F_{phys.}/F_0 = F_{phys.}/\sim$$

states differing only by terms with α_L^+ are equivalent

Note: Since $A_\mu \to A_\mu + \partial_\mu \alpha$ corresponds to $\tilde{A}_\mu \to \tilde{A}_\mu + ik_\mu \tilde{\alpha}$ in Fourier space, the freedom of adding states from F_0 (recall $\epsilon_L \sim k$) corresponds to the "residual gauge freedom" of the classical theory. ("residual" since, thanks to $k^2=0$, $k.\tilde{A}=0$ is satisfied before & after this gauge trf.)

Note: F, F_{phys}, $\mathcal{H}$ were defined abstractly and the "frame choice" or choice of a vector n was only used for the explicit construction. Hence we can be sure that our result does not depend on n.

8.3 The photon propagator

$$\langle 0|A_\mu(x)A_\nu(y)|0\rangle = \langle 0|\int d\tilde{k}\,d\tilde{k}'\,e^{-ikx+ik'y}\,a_{\vec{k},\mu}\,a^+_{\vec{k}',\nu}|0\rangle$$

only a, only a^+ are relevant

$$= -\eta_{\mu\nu}\int d\tilde{k}\,e^{-ik(x-y)}$$

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \Theta(x^0-y^0)\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \{x\leftrightarrow y\}$$

$$= \Theta(x^0-y^0)\left(-\eta_{\mu\nu}\int d\tilde{k}\,e^{-ik(x-y)}\right) + \{x\leftrightarrow y\}$$

$$= -\eta_{\mu\nu}\left[\Theta(x^0-y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \{x\leftrightarrow y\}\right]$$

$$(m^2=0)$$

$$= -\eta_{\mu\nu}\langle 0|T\varphi(x)\varphi(y)|0\rangle = \underline{\underline{-\eta_{\mu\nu}D_F(x-y,m^2=0)}}$$

- A fact which will be derived with ease in the path integral approach (see also problems):

For more general gauge choice ($\lambda\neq 1$),

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int\frac{d^4k}{(2\pi)^4}(-i)\left(\frac{\eta_{\mu\nu}}{k^2+i\varepsilon}+\frac{1-\lambda}{\lambda}\cdot\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right)\cdot e^{-ik(x-y)}$$

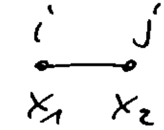
8.4 Feynman rules for scalar QED

- As a warm-up, consider first a model with N different real scalars:

$$\mathcal{L} = \sum_{i=1}^{N} \frac{1}{2}\left((\partial\varphi^i)^2 - m^2(\varphi^i)^2\right) - \frac{\lambda}{8}\left(\sum_{i=1}^{N}(\varphi^i)^2\right)^2$$

This particular form of quartic interaction respects the $O(N)$-symmetry of the free theory.

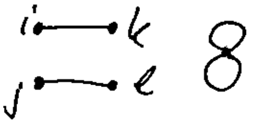
- It is immediately clear that

$$\langle T\varphi_1^i\varphi_2^j\rangle \equiv \varphi_1^i\varphi_2^j = \delta^{ij} D_F(x_1-x_2) =$$ 

- The vertex is derived by considering

$$\langle T\varphi_1^i\varphi_2^j\varphi_3^k\varphi_4^l \int d^4x\left(\frac{-i\lambda}{8}\right)\left(\delta_{mn}\varphi_x^m\varphi_x^n\right)\left(\delta_{pq}\varphi_x^p\varphi_x^q\right)\rangle$$

- We consider only the connected part, i.e.

and <u>not</u> 

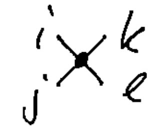
- This means that $\varphi_1, \varphi_2, \varphi_3$ and φ_4 must each be contracted with one of the φ_x. Contracting according to $\varphi_1^i\varphi_x^m\ \varphi_2^j\varphi_x^n\ \varphi_3^k\varphi_x^p\ \varphi_4^l\varphi_x^q$ we clearly get

$$-\frac{i\lambda}{8}\delta^{ij}\delta^{kl} D_F(x_1-x)\, D_F(x_2-x)\, D_F(x_3-x)\, D_F(x_4-x)$$

and hence a contribution $\frac{-i\lambda}{8}\delta^{ij}\delta^{kl}$ to the vertex.

- The exact same contribution arises by exchanging φ_x^m & φ_x^n ; φ_x^p & φ_x^q ; $(\varphi_x^m\varphi_x^n)$ & $(\varphi_x^p\varphi_x^q)$ $\Rightarrow$ factor $1/8$ disappears

- One can think of this contribution as of choosing "pairing up" i with j & k with l. Clearly, two truly different such pairings exist: $(ik)(jl)$ and $(il)(jk)$. Hence:



$$= -i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right).$$

(As a check, note that $8\cdot 3 = 4!$, so we didn't forget anything.)

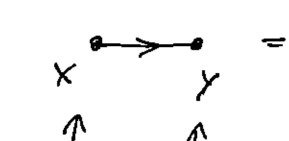
- As a further warm-up exercise, consider a single complex scalar. From

$$\phi(x) = \int d\tilde{k}\left(a^+_{\bar{k}}\, e^{ikx} + b^-_{\bar{k}}\, e^{-ikx}\right)$$

we immediately conclude:

$$\overbracket{\phi_x \phi_y} = 0 \;;\quad \overbracket{\phi_x \phi_y^+} = \overbracket{\phi_x^+ \phi_y} = D_F(x-y).$$

- Since ϕ is different from ϕ^+, we can now assign a direction to the corresponding line in the Feynman rule:



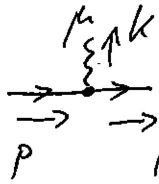
$= D_F(x-y)$

(Here the arrow gives the direction in which the b-particle propagates.)

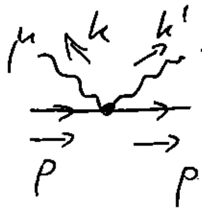
- After these preliminaries, we simply state the Feynman rules of scalar QED. We will then give a partial derivation, which you can complete yourself in the tutorials & independently.

Scalar QED:

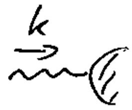
$\longrightarrow \;=\; \dfrac{i}{k^2-m^2+i\varepsilon}$ / ${}_\mu\!\sim\!\sim\!\sim_\nu \;=\; \dfrac{-i\eta^{\mu\nu}}{k^2+i\varepsilon}$



$= -ie\,(p_\mu+p'_\mu)$ with $p' = p-k$



$= 2ie^2\eta^{\mu\nu}$ with $p' = p-k-k'$

 $= Z_\phi^{1/2}$ (ext. scalar) /  $= Z_A^{1/2}\cdot \varepsilon_\mu(k)$ (incoming photon)

& same with $\varepsilon \to \varepsilon^*$ for the outgoing photon

- A proper derivation requires, in principle, to go through the whole procedure of the last sections (free fcts., LSZ etc.) with our new theory replacing the real scalar $\lambda\varphi^4$-model.
- Simple example, illustrating the main points: Consider

$\gamma+\phi\to\phi$, i.e.

(ignore off-shellness of outgoing ϕ)

$$\Rightarrow \langle 0|\, b_{\bar p'} \left(i\int d^4x\, \mathcal{L}_{int} \right) b^+_{\bar p}\, \varepsilon^\mu(k)\, a^+_{\bar k,\mu}\, |0\rangle = i\mathcal{M}_{fi}\,(2\pi)^4\delta^4(\ldots)$$

$\uparrow$

cubic part of $|D_\mu\phi|^2$, i.e.

$$-ie\,A^\nu\phi^+\partial_\nu\phi \quad + \text{ second similar term}$$

$\int d\tilde q\, a^\nu_{\bar q}\, e^{-iqx}$ (acts on $a^+_{\bar k,\mu}|0\rangle$) $\qquad -ip_\nu \qquad \int d\tilde q'\, b_{\bar q'}\, e^{-iq'x}$ (acts on $b^+_{\bar p}|0\rangle$)

Note: $a^{\nu}_{\bar{q}} a^{+\mu}_{\bar{k}} = -\eta^{\mu\nu} (2\pi)^3 2k_0 (\bar{k}-\bar{q})$

Collecting everything find:

$$i\mathcal{M}_{fi} = -iep^{\mu}\varepsilon_{\mu}(k) \underbrace{- iep'^{\mu}\varepsilon_{\mu}(k)}_{\text{from second term}} = \overbrace{-ie(p^{\mu}+p'^{\mu})}^{\text{vertex}}\underbrace{\varepsilon_{\mu}(k)}_{\text{incom. photon}}$$

• Next exercise: term $\sim A^{\mu}A_{\mu}\phi\phi^{+}$ gives 2γ-2ϕ-vertex ...

<u>Important comment:</u>

Since interactions involve $\dot{A}$, the canonical momentum of A receives a contribution from $\mathcal{L}_{int}$. Thus, it is not true any more that $\mathcal{H}_{int} = -\mathcal{L}_{int}$ and, since $\mathcal{H}_{int}$ is the crucial quantity in pert. theory, our derivation above is not correct. However, at the same time, it is too naive to assume that ∂_{μ} commutes with contractions. For example

$$\langle T\, \partial_{\mu}\varphi_x \partial_{\nu}\varphi^{+}_{y} \rangle = \frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}} \langle T \varphi_x \varphi_y \rangle - i\eta_{\mu 0}\eta_{\nu 0}\delta^4(x-y).$$

This effect also corrects the Feynman rules precisely compensating the "error" we made by assuming $\mathcal{H}_{int} = -\mathcal{L}_{int}$. In fact, this <u>had</u> to be the case to ensure that the final result is Poinc.-invar. In summary, our naive analysis gave the correct result. For details see Itzykson/Zuber, Sect. "Scalar Electrod."