9 Spinors

9.1 Fields and representations

- We already know 3 types of fields transforming differently under the Lorentz group:

$$\varphi(x) \longrightarrow \varphi(\Lambda^{-1}x)$$
$$A^\mu(x) \longrightarrow \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$
$$F^{\mu\nu}(x) \longrightarrow \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda^{-1}x),$$

the last one not being elementary (but that's not important at the moment). Clearly, we could consider even higher tensors.

- If we define a field as a map $\mathbb{R}^4 \longrightarrow V$

$$x^\mu \longmapsto \{\phi^i(x)\},$$

we see that the Lorentz group acts in two ways:
  1) on the argument (or on $\mathbb{R}^4$) – alway the same
  2) on the field value (or on V) – different from field to field.

- More formally, a field is characterized by a vector space V and a repres. of $SO(1,3)$ on V.

<u>Reminder</u>: For any group G a repr. is a map $G \xrightarrow{R} GL(V)$ (gen. lin. tfs on V) such that $R(\mathbb{1}) = \mathbb{1}$; $R(g\cdot h) = R(g)\cdot R(h)$

- Our examples above have:

scalar: $V = \mathbb{R}$ or $\mathbb{C}$; $R(\Lambda) = 1$ (trivial repr.)
vector: $V = \mathbb{R}^4$; $R(\Lambda) = \Lambda$ (fundamental repr.)
tensor: $V = \mathbb{R}^4 \otimes \mathbb{R}^4$; $R(\Lambda) = \Lambda \otimes \Lambda$ (antisymm. tensor)

[$F^{\mu\nu}$ or, more correctly $F^{\mu\nu}\hat{e}_{(\mu)} \otimes \hat{e}_{(\nu)}$, is in antisymm. subspace]

- Note also: If we want to fit the tensor (in our case the antisymm.-rank-2-tensor) in the general $\{\phi^i(x)\}$-notation, then the index i runs over all pairs of distinct indices μ & ν : $\{\mu\nu\} \hat{=} i$

$\Rightarrow$ "our repr. has $(4\times 4-4)/2 = \binom{4}{2} = 6$ dims."

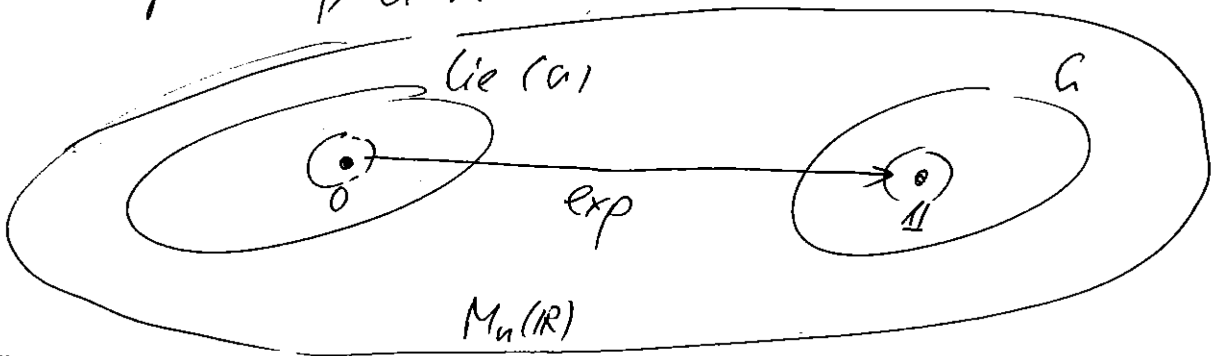
9.2 Some remarks on Lie groups

(Not replacing a proper course on Lie groups & their repr.s!)

- Lie groups are groups which are also manifolds, such that group operations are diffeomorphisms. (If you don't yet know what a manifold is think of smooth subspaces of $\mathbb{R}^n$ with the group operation being differentiable.)
- The prime examples are matrix groups like $O(n)$ (orthogonal); $U(n)$ (unitary); $Sp(n)$ (symplectic).
- With a Lie group G always comes a Lie algebra $\mathrm{Lie}(G)$.
- A Lie alg. is a vector space $\mathfrak{g}$ with a bilinear, antisymm. map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, $(a,b)\mapsto[a,b]$ satisfying the Jacobi identity: $[a,[b,c]]+[c,[a,b]]+[b,[c,a]]=0$.
- We will only need the case of matrix groups where the relation between Lie alg. & Lie group can be understood elementary: (But even here: no proofs!)
  - The map "exp" is a diffeom. of small neighborhoods of 0 & $\mathbb{1}$ in $M_n(\mathbb{R})$ & $GL(n,\mathbb{R})$:

$$\boxed{\underbrace{\exp(a)=g}\;;\;\exp(0)=\mathbb{1} \quad a \text{ near } 0;\ g \text{ near } \mathbb{1}.}$$

- $lie(G)$ is the lin. subspace of $M_n(\mathbb{R})$ generated by $\exp^{-1}(O_{\mathbb{1}})$, where $O_{\mathbb{1}}$ is a neighb.hood of $\mathbb{1} \in G \subset GL(n)$.
- In other words: $\exp(a) = g$ with $a \in lie(G)$; $g \in G$ maps (at least) a small patch of $lie(G)$ near 0 to (a small patch of) G near $\mathbb{1}$.



Example: $G = SO(3)$; $lie(G) = \{$ antisymmetric 3×3 matrices $\}$

Indeed: If $R = \exp(T)$, then $RR^T = \exp(T)\exp(T)^T = \ldots$
$\ldots = \exp(T)\exp(-T) = \mathbb{1}$ because T is antisymm.
(The "S" of "SO" is not visible at the Lie-alg. level.)

- It is illuminating to see in general that, if $a, b \in lie(G)$, then $\exp[a,b] \in G$: (now $[\,,\,]$ is a simple commutator!)

Let $A = \exp(\varepsilon a)$; $B = \exp(\varepsilon b)$.
Obviously, $ABA^{-1}B^{-1} \equiv C \in G$.
Thus, $C = \left(\mathbb{1} + \varepsilon a + \frac{\varepsilon^2}{2} a^2\right)\left(\mathbb{1} + \varepsilon b + \ldots\right)(\ldots)(\ldots) + O(\varepsilon^3)$

$$= \mathbb{1} + \varepsilon^2 [a,b] + O(\varepsilon^3) = \exp(\varepsilon^2 [a,b]) + O(\varepsilon^3)$$

$$C^{1/\varepsilon^2} = \exp[a,b] \quad \checkmark$$

- In analogy to groups, we also have the concept of a repr. of a Lie-alg.:

$$\begin{aligned} Lie(G) &\xrightarrow{R} gl(V)^* \\ a &\longmapsto R(a) \end{aligned}$$

with: $R(0)=0$; $R([a,b]) = [R(a), R(b)]$.

$*M_n(\mathbb{R})$ for matrices

Note: V is in general not the same vector space as the $\mathbb{R}^n$ underlying $G \subset GL(n)$ in the matrix group case.

Crucial fact:

Given some repr. R of $Lie(G)$, we can always construct an associated repr. of G (which we will also call R by abuse of notation), such that

$$R(A) = \exp(R(a)) \quad \text{if} \quad A = \exp(a)$$

Sketch of proof: We can take the above "feature" of R as its definition, at least near $\mathbb{1}$:

$$R(A) \equiv \exp(R(\exp^{-1}(A))).$$

We then need to show that this is really a repr. of G, i.e. $R(A)R(B) = R(AB)$. Thus, given $A \equiv e^a$, $B \equiv e^b$, $AB \equiv C \equiv e^c$ we need to show

$$e^{R(a)} e^{R(b)} = e^{R(c)}.$$

We already know that

$$e^a e^b = e^c$$

and $e^a e^b = e^{z(a,b)}$ with

$\Rightarrow z(a,b) = c$

$$z(a,b) = a + b + \tfrac{1}{2}[a,b] + \tfrac{1}{12}[a,[a,b]] - \tfrac{1}{12}[b,[a,b]] + \dots$$

(Baker–Campbell–Hausdorff; crucial: just commutators!)

This implies

$$e^{R(a)}e^{R(b)} \underset{\text{BCH}}{=} e^{Z(R(a),R(b))} \underset{\substack{\text{feature of Lie-alg.}\\ \text{representation}}}{=} e^{R(Z(a,b))} \underset{\text{see above}}{=} e^{R(c)} \quad \checkmark$$

9.3 The spinor repr. of $SO(1,3)$

- Let's first understand $\mathrm{Lie}(SO(1,3)) \equiv so(1,3)$:

$$\underset{\substack{\text{"physicists}\\ \text{convention"}}}{\Lambda = e^{iT}} \xrightarrow[\text{"small" } T]{\text{apply to}} \Lambda_\mu{}^\nu = \delta_\mu{}^\nu + i T_\mu{}^\nu .$$

- See what $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$ implies for T:

$$(\delta_\mu{}^\nu + iT_\mu{}^\nu)(\delta_\rho{}^\sigma + iT_\rho{}^\sigma)\eta_{\nu\sigma} = \eta_{\mu\rho} + O(T^2)$$

$$\boxed{T_{\mu\rho} + T_{\rho\mu} = 0}$$

(after lowering the 2nd index, the $SO(1,3)$-generators are antisymm. matrices)

- Canonical basis: $(M_{\rho\sigma})_\mu{}^\nu$, such that

$$T_\mu{}^\nu = t^{\rho\sigma}(M_{\rho\sigma})_\mu{}^\nu$$

both antisymm. in $\rho\sigma$
$\Rightarrow$ 6 generators.

<u>Problem</u>: Explicitly define a basis of matrices $\{M_{\mu\nu}\}$ such that each $M_{\mu\nu}$ generates rotations in the μ-ν-plane and

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}).$$

- Any $\Lambda \in SO^+(1,3)$ (i.e. in the identity component) can be written as $\Lambda = \exp(i t^{\mu\nu} M_{\mu\nu})$. [w/o proof]
- To define the spinor repr., we first introduce the <u>Clifford algebra</u> as the abstract algebra generated by $\mathbb{1}$ & 4 elements γ^μ which satisfy
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\cdot\mathbb{1}.$$

[Here $\{a,b\}$ is the "anti-commutator" $ab + ba$ and we will often suppress the "$\mathbb{1}$" on the r.h. side.]

- We will see that this algebra is finite dimensional.
- Even though a lot more could be done "abstractly", we immediately give an explicit repr. in terms of 4×4 matrices:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \quad \text{where} \quad \sigma^\mu \equiv (\sigma^0, \sigma^i) \equiv \left\{ \mathbb{1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$
$$\& \quad \bar\sigma^\mu \equiv (\sigma^0, -\sigma^i).$$

- We will also use $\gamma_\mu \equiv \eta_{\mu\nu}\gamma^\nu$.
- To check that these γ's indeed represent the abstract Cliff. algebra, write

$$\{\gamma^\mu, \gamma^\nu\} = \left\{ \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix} \right\} = \begin{pmatrix} \sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

and separately analyse the cases $\mu,\nu = 0,0 \,/\, 0,i \,/\, i,j$.
Use also $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\cdot\mathbb{1}$.

- <u>Problem</u>: Show that $\mathcal{M}_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ satisfy the same commut. relations as the $M_{\mu\nu}$ introduced earlier.

- Thus, the $\mathcal{M}_{\mu\nu}$ represent $so(1,3)$ and we can construct a corresponding repr. of $SO(1,3)$, at least near $\mathbb{1}$:
  - Write $\Lambda \in SO(1,3)$ as $\Lambda = \exp(i t^{\mu\nu} M_{\mu\nu})$.
  - Define action on $\mathbb{C}^4$ as

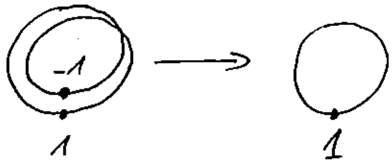
$$\psi_D \overset{\Lambda}{\longmapsto} S(\Lambda)\cdot\psi_D \;; \quad \boxed{\begin{aligned} S(\Lambda) &= \exp(i t^{\mu\nu}\mathcal{M}_{\mu\nu}) \\ &= \exp\left(i t^{\mu\nu}\left(\tfrac{i}{4}[\gamma_\mu,\gamma_\nu]\right)\right)\end{aligned}}$$

- A "Dirac spinor" is a set of fields $(\psi_D)_a(x)$, $a = 1..4$, transforming as

$$(\psi_D)_a(x) \xrightarrow{\;\Lambda\;} S(\Lambda)_a{}^b (\psi_D)_b(\Lambda^{-1}x).$$

<u>Note:</u> The map $\Lambda \to S(\Lambda)$ is not defined globally on $SO^+(1,3)$. To see this, pick some rotation axis and call the coresp. generator $T \equiv t^{\mu\nu}M_{\mu\nu}$. Clearly, the rotation is given by $\Lambda(\varphi) = \exp(i\varphi T)$ and $\Lambda(2\pi) = \mathbb{1}$. However, for $T_S \equiv t^{\mu\nu}\mathcal{M}_{\mu\nu}$ one finds $S(2\pi) = \exp(i\varphi T_S) = -\mathbb{1}$. The resolution is simple:

The group $Spin(1,3)$ [$\equiv$ the group generated by the $\mathcal{M}_{\mu\nu}$'s] is the fundamental symm. of nature. The map $\mathcal{M}_{\mu\nu} \to M_{\mu\nu}$ leads to an associated repr. of this group acting on vectors: $\Lambda = \Lambda(S)$. Spin$(1,3)$ is the "double cover" of $SO^+(1,3)$. Visualize:



(Some more details will be given below.)

- The repr. of $SO(1,3)$ (more correctly of $Spin(1,3)$) on Dirac spinors provided above is not irreducible:

$$\mathcal{M}_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] = \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_\nu \\ \bar{\sigma}_\nu & 0 \end{pmatrix}\right]$$

$$= \frac{i}{4}\begin{pmatrix} \sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu & 0 \\ 0 & \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu \end{pmatrix}.$$

is block diagonal!

$\Rightarrow$ We can write $\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$, where α & $\dot{\alpha}$ run over 1, 2,

with the Weyl spinor ψ (& the compl. conjugate Weyl spinor $\bar{\chi}$) transforming independently.

This is not obvious at the moment.

- The above decomposition of ψ_D can also be defined abstractly (i.e. w/o using our explicit representation of the γ's):
  - Introduce $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$
  - Obviously, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$. Hence $\gamma^5\mathcal{M}_{\mu\nu} = \mathcal{M}_{\mu\nu}\gamma^5$.
  - Easy to check: $(\gamma^5)^2 = \mathbb{1}$.
  - Define $P_L \equiv \frac{1}{2}(\mathbb{1} - \gamma^5)$; $P_R \equiv \frac{1}{2}(\mathbb{1} + \gamma^5)$
  - Easy to check: $P_L^2 = P_L$; $P_R^2 = P_R$; $P_L + P_R = \mathbb{1}$
  - These properties make P_L & P_R projection operators. They induce a decomposition of the space on which they act as $V = V_L \oplus V_R \equiv \text{Im}(P_L) \oplus \text{Im}(P_R)$.
  - It follows that $\psi_{D,L} = P_L\psi_D$ & $\psi_{D,R} = P_R\psi_D$ transform independently (since $P_{L,R}$ commute with $\mathcal{M}_{\mu\nu}$).

- We will call $\psi_{D,L}$ & $\psi_{D,R}$ l.h. & r.h. Dirac spinors.
- Since in our explicit repr.

$$\gamma^5 = \begin{pmatrix} -1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} \; ; \; P_L = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 0 \end{pmatrix} \; ; \; P_R \begin{pmatrix} 0 & 0 \\ 0 & 1\!\!1 \end{pmatrix},$$

we have

$$\psi_{D,L} = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \; ; \; \psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix}.$$

(The phys. information in each of them is equiv. to that in a Weyl spinor.)

Note: All of the above works in almost complete analogy in any <u>even</u> dimension d. The Dirac spinor has dim. $2^{d/2}$. For d odd, one uses the Cliff. alg. with $(d-1)\,\gamma$'s, as above, and adds $\gamma^d \sim \gamma^0\gamma^1 \cdots \gamma^{d-2}$, which is now <u>not</u> special compared to the other γ's. It does not commute with $M_{\mu\nu}$ and L/R or Weyl spinors don't exist.
(→ see Polchinski, "String theory", vol. II, Appendix for more on "Spinors in various dimensions")

- An interesting & useful fact <u>special</u> to $d=4$:

$$\boxed{Spin(1,3) = SL(2,\mathbb{C})}$$

↑ 2×2 matrices w/ det = 1.

- Using this, the <u>$2\to1$ map to $SO(1,3)$</u> can be given very explicitly:

Let $M \in SL(2,\mathbb{C})$; $v \in \mathbb{R}^4$; $\hat{v} \equiv v_\mu \sigma^\mu$.
(Since $\{\sigma^\mu\}$ are a basis of hermitian 2×2 matrices, $\hat{v}$ is a generic 2×2 matrix for generic v.)

Define $\hat{v}' \equiv M\hat{v}M^+$. Define v' by $\hat{v}' = v'_\mu \sigma^\mu$.
Calculate $(v')^2$:

$$(v')^2 = (v'^0)^2 - (\vec{v}\,')^2 = \det\begin{pmatrix} v_0'+v_3' & v_1'-iv_2' \\ v_1'+iv_2' & v_0'-v_3' \end{pmatrix} = \det \hat{v}'$$

$$= \det \hat{v} = \det\begin{pmatrix} v_0+v_3 & v_1-iv_2 \\ v_1+iv_2 & v_0-v_3 \end{pmatrix} = v^2 .$$

$\Rightarrow$ Any $M \in SL(2,\mathbb{C})$ defines a map $\hat{v} \to M\hat{v}M^+ \equiv \hat{v}'$ on herm. 2×2 matrices and hence a map $v_\mu \to v_\mu'$ preserving the length.

$\Rightarrow$ $\exists\ \Lambda = \Lambda(M) \in SO(1,3)$ s.t. $v_\mu' = \Lambda_\mu{}^\nu v_\nu$.

(Obviously, $\Lambda(M) = \Lambda(-M)$. Hence the map is "$2 \to 1$".)

Fact: Our Weyl spinor ψ_α transforms as $\psi_\alpha \to M_\alpha{}^\beta \psi_\beta$. The "other" two-comp. spinor is $\bar{\chi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}$ with $\bar{\chi}_{\dot\beta}$ transforming as $\bar{\chi}_{\dot\beta} \to \bar{M}_{\dot\beta}{}^{\dot\alpha}\bar{\chi}_{\dot\beta}$ (compl. conj. of $\chi_\beta = M_\beta{}^\alpha \chi_\alpha$). [In principle, you can check all this using our def. $\psi_D = \binom{\psi}{\bar\chi}$.]

"$SL(2,\mathbb{C})$ is the true symm. group of space-time."
(+ translations, of course...)

Note: The relation between $SU(2)$ & $SO(3)$, underlying the existence of the 2-comp. non-relativistic spinor of QM, is completely analogous.

9.4 Invariants involving spinors, lagrangian, EOMs

- We now focus exclusively on Dirac spinors and drop the index "D" for brevity: $\psi_D \to \psi$.
- Clearly, to write down lagrangians with ψ we need to construct invariants (Lorentz singlets) with ψ.

(1) • Recall that for any unitary repr. of a symm. group, $v \to Uv$, $U \in \mathcal{U}(n)$, the object $v^+v \equiv \sum_i \bar{v}^i v^i$ would be invariant: $v'^+v' = (Uv)^+Uv = v^+U^+Uv = v^+v$.

Infinitesimally, this follows from

$$v'^+v' \simeq ((1+iT)v)^+(1+iT)^+v \simeq v^+(1-iT^+)(1+iT)v$$
$$\simeq v^+(1+i(T-T^+))v$$

and $T = T^+$.

• In our case, $\psi \to (1+it^{\mu\nu}\mathcal{M}_{\mu\nu})\psi$. From $(\gamma^0)^+ = \gamma^0$ & $(\gamma^i)^+ = -\gamma^i$ we conclude

$$\mathcal{M}_{0i}^+ = -\mathcal{M}_{0i}, \quad \mathcal{M}_{ij}^+ = \mathcal{M}_{ij},$$

such that $\psi^+\psi$ is <u>not</u> invariant.

(<u>Note</u>: $SO(1,3)$ is non-compact, implying that no finite-dimensional unitary repr.s exist.)

• However, it is easy to see that $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+$

& $\gamma^0\mathcal{M}_{\mu\nu}^+\gamma^0 = \mathcal{M}_{\mu\nu}$.

• Hence $\psi^+\gamma^0 \to \psi^+(1+it^{\mu\nu}\mathcal{M}_{\mu\nu})^+\gamma^0$

$$= \psi^+(1-it^{\mu\nu}\mathcal{M}_{\mu\nu}^+)\gamma^0 = \psi^+\gamma^0(1-it^{\mu\nu}\mathcal{M}_{\mu\nu}).$$

$\Rightarrow$ $\psi^+\gamma^0\psi$ is invariant.

• We define $\bar{\psi} \equiv \psi^+\gamma^0$ and call this invariant $\underline{\underline{\bar{\psi}\psi}}$.

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(2) • Using the last problem sheet, it is an easy excercise or "corollary" to show that:

$$[\mathcal{M}_{\mu\nu}, \gamma_\rho] = -(M_{\mu\nu})_\rho{}^\sigma \gamma_\sigma$$

• It follows that

$$(1+i t^{\mu\nu} \mathcal{M}_{\mu\nu})\gamma_\rho (1-i t^{\mu\nu}\mathcal{M}_{\mu\nu}) = (1-i t^{\mu\nu}M_{\mu\nu})_\rho{}^\sigma \gamma_\sigma$$

or, after exponentiation $\left[(1+\frac{a}{n})^n \simeq e^a\right]$

$$S(\Lambda)\gamma_\rho S(\Lambda)^{-1} = (\Lambda^{-1})_\rho{}^\sigma \gamma_\sigma .$$

• Multiplying by Λ and making spinor indices explicit gives

$$\Lambda_\rho{}^\sigma S(\Lambda)_a{}^b (\gamma_\sigma)_b{}^c (S(\Lambda)^{-1})_c{}^d = (\gamma_\rho)_a{}^d .$$

• We have learned that $(\gamma_\sigma)_a{}^b$ is an <u>invariant tensor</u> of $SO(1,3)$, with σ – vector index

a – spinor index (like ψ_a)

b – (upper or inverse) spinor index (like $\bar\psi^b$).

• We also know that

$\bar\psi\gamma^\mu\psi$ is a vector and hence $\bar\psi\gamma^\mu\psi v_\mu$ (with v a vector) is <u>an invariant.</u>

$\Rightarrow$ $\boxed{\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi}$ is our lowest-order (in fields & derivatives) Lagrangian.

<u>Problem</u>: Show that the "i" is needed for S to be real.

• EOMs are derived by treating ψ, $\bar\psi$ as indep. variables (like ϕ, ϕ^*):

$$\delta S = \int d^4x\, \delta\mathcal{L} = \int d^4x \left[\delta\bar\psi(i\underbrace{\not\partial}_{\equiv\gamma^\mu\partial_\mu} - m)\psi) + \bar\psi(i\underbrace{\overleftarrow{\not\partial}}_{\equiv\gamma^\mu\overleftarrow{\partial}_\mu} - m)\delta\psi\right]$$

$\Rightarrow$ $\boxed{(i\not\partial - m)\psi = 0}$ <u>Dirac eq.</u> (The other eq. is the herm. conj.)

- Important fact: ψ solves Dirac-eq. $\Rightarrow$ ψ solves Klein-Gordon-eq.
- Demonstration: $0 = (-i\not\partial - m)(i\not\partial - m)\psi = (\not\partial^2 + m^2)\psi$

(Note: for any vector p we have $\not p^2 = \gamma_\mu \gamma_\nu p^\mu p^\nu$
$= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)p^\mu p^\nu = \eta_{\mu\nu} p^\mu p^\nu = p^2$)

$\Rightarrow (\partial^2 + m^2)\psi = 0$.

9.5 Solutions of the Dirac eq.

Ansatz: $\psi(x) = u(p) e^{-ipx}$ with $p^2 = m^2$; $p_0 > 0$.

$(i\not\partial - m)\psi = 0 \quad \Rightarrow \quad (\not p - m)u(p) = 0$

- Choose frame where $p = (m, \vec{0}) \Rightarrow m(\gamma^0 - \mathbb{1})u(p) = 0$

$\Rightarrow \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p) = 0$

- Ansatz (compl. general!): $u(p) = \begin{pmatrix} \xi \\ \xi' \end{pmatrix} \Rightarrow \xi - \xi' = 0$

$\Rightarrow \xi = \xi'$. Hence we have two indep. sol.s., which we can write, e.g., as: $u_s \sim \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$ with $s = 1, 2$

& $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- It will be convenient to choose the following normalization:

$$\boxed{u_s(p) \equiv \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \text{ in frame where } p = (m, \vec{0})}$$

(Since we know how a spinor transforms, this defines $u_s(p)$ in complete generality, i.e. for any p.)

Comment: • Staying in the rest frame ($p = (m, \vec{0})$), the relation to the non-rel. spinor of QM is particularly obvious:

- Indeed, if we exclude boosts, we are in
$$SO(3) \subset SO(1,3),$$
corresponding to the restriction
$$t^{\mu\nu}\mathcal{M}_{\mu\nu} \to t^{ij}\mathcal{M}_{ij} = t^{ij}\cdot\frac{i}{4}\begin{pmatrix} \sigma_i\bar\sigma_j - \sigma_j\bar\sigma_i & 0 \\ 0 & \bar\sigma_i\sigma_j - \bar\sigma_j\sigma_i \end{pmatrix}$$
$$= t^{ij}\cdot\frac{1}{2}\begin{pmatrix} \varepsilon_{ijk}\sigma_k & 0 \\ 0 & \varepsilon_{ijk}\sigma_k \end{pmatrix}.$$
$$\left(\text{using } [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k\right)$$

- We see that both upper & lower two-component spinor rotate as in QM. Even more explicitly, for a rotation by φ around the 3-axis we must pick
$$t^{ij} = \tfrac{1}{2}\varepsilon^{ijk}(\hat e_3)^k\cdot\varphi\,,$$
and we find
$$\exp(it^{ij}\mathcal{M}_{ij}) = \begin{pmatrix} \exp(i\varphi\frac{1}{2}\sigma_3) & 0 \\ 0 & \exp(i\varphi\frac{1}{2}\sigma_3) \end{pmatrix}.$$

- This is consistent with $SU(2) \subset SL(2,\mathbb{C})$ and the $SL(2,\mathbb{C})$ action on spinors described earlier. It also shows explicitly that our Dirac spinors correspond to the familiar spin-1/2 states of QM.

- A second set of so-called "neg.-frequency" solutions also exists.:
$$\psi(x) = v(p)\,e^{+ipx} \qquad (p^2 = m^2,\ p_0 > 0).$$

- As above,
$$(i\not\partial - m)\psi = 0 \quad \Rightarrow \quad (\not p + m)\,v(p) = 0$$

- Choosing $p = (m, \vec 0)$: $\quad \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}\cdot v(p) = 0$

$\Rightarrow \quad v_s(p) = \sqrt{m}\begin{pmatrix}\eta_s\\ -\eta_s\end{pmatrix}, \quad s=1,2, \quad \eta_1 = \begin{pmatrix}1\\0\end{pmatrix}, \eta_2 = \begin{pmatrix}0\\1\end{pmatrix}.$

- Our basis choice & normalization were such that:

$$\left\| \begin{array}{ll} \bar{u}_r(p)u_s(p) = 2m\,\delta_{rs} & \bar{u}_r(p)v_s(p) = 0 \\ \bar{v}_r(p)v_s(p) = -2m\,\delta_{rs} & \bar{v}_r(p)u_s(p) = 0 \end{array} \right\|$$

[Once established in the rest frame, this holds in all frames by Lorentz-covariance.]

- In addition to the "orthonormality" relations above, we also have a 'kind of' completeness relations:

$$\left\| \begin{array}{l} \sum_{s=1}^{2} (u_s(p))_a (\bar{u}_s(p))^b = (\not{p}+m)_a{}^b \\ \sum_s v_s(p)\bar{v}_s(p) = \not{p}-m \quad \Leftarrow \text{simplifying matrix notation} \end{array} \right\|$$

<u>Derivation</u>: We demonstrate the above equality of matrices by letting both l.h. & r.h. side act on the basis $\{u_s(p), v_s(p)\}$ of the 'spinor space' $\mathbb{C}^4$:

$$\left.\begin{array}{l} \left(\sum_s u_s(p)\bar{u}_s(p)\right)\cdot u_r(p) = \sum_s u_s(p)\cdot 2m\,\delta_{rs} = 2m\,u_r(p) \\ \left(\sum_s u_s(p)\bar{u}_s(p)\right)\cdot v_r(p) = 0 \end{array}\right\} \text{using orthon.}$$

$$\left.\begin{array}{l} (\not{p}+m)u_r(p) = (\not{p}-m+2m)u_r(p) = 2m\,u_r(p) \\ (\not{p}+m)v_r(p) = 0 \quad \checkmark \end{array}\right\} \text{using the definition of } u \,\&\, v.$$

The proof of $\sum_s v_s(p)\bar{v}_s(p) = \not{p}-m$ proceeds analogously.

<u>Final comment</u>: It is easy to memorize the signs in the above relations by consistency with the Dirac eq. $((\not{p}-m)u(p)=0)$:

$(\not{p}-m)\left(\sum u_s(p)\bar{u}_s(p)\right) = (\not{p}-m)(\not{p}+m) \Rightarrow 0 = p^2-m^2 \quad \checkmark.$