Quantum Field Theory

1 Introduction

1.1 General Idea

- Starting point: Poincare invariant classical field theory defined by an action

- Familiar example: Electrodynamics $(c = 1)$

$$S[A_\mu] = \int d^4x \, \mathcal{L}(A_\mu(x), \partial_\mu A_\nu(x))$$

$$\left(\mu = 0,1,2,3 \;;\; \partial_\mu = \frac{\partial}{\partial x^\mu} \;;\; x^0 = t \;;\; \{x^i\} = \bar{x} \;;\; i = 1,2,3\right)$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;;\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} F_{\rho\sigma}$$

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) = \eta^{\mu\nu}$$

[more generally: $g_{\mu\nu}$ – metric ; $g^{\mu\nu}$ – inverse metric ; in flat space: $g_{\mu\nu} = \eta_{\mu\nu}$]

<u>Note</u>: The convention $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is also frequently used.

- Interpret $S = \int d^4x \, \mathcal{L} = \int dt \, L \;;\; L = \int d^3x \, \mathcal{L}$

as action for many variables $A_\mu(t, \bar{x})$.

labels for different variables

(Think of space as modelled by an infinitely dense lattice, with one $A_\mu(t)$ for every lattice point.)

- From this perspective, we are in the framework of usual lagrangian mechanics (with many variables); next, we pass to the hamiltonian formulation

$$q_i \mathrel{\hat{=}} A_\mu(\bar{x}) \quad ; \quad p_i \mathrel{\hat{=}} \dot{A}_\mu(\bar{x}) \; ,$$

and quantize:
$$q_i \, , p_i \;\; \longrightarrow \;\; \hat{q}_i \, , \hat{p}_i \; .$$

- We will construct a Hilbert space on which the operators $\hat{q}_i$, $\hat{p}_i$ act. It will contain a vacuum $|0\rangle$ and excited states (as familiar from the quantum harmonic oscillator). The latter correspond to states with one, two, three, ... photons in the vacuum. We will introduce interactions which will allow these photons to scatter off each other.

- This procedure works for scalar, vector, spinor fields and describes, in particular, modern collider experiments (like the LHC). More generally, all presently known fundamental interactions (with certain limitations in the case of gravity) fit into this framework.

1.2 Topics to be covered (in two terms)

- Quantization of free scalar field ; Noether theorem
- Perturbation theory ($\lambda\varphi^4$-theory, cross sections, LSZ-formalism, Wick theorem, Feynman rules)
- Abelian vector fields / Electrodynamics (Gupta-Bleuler-Quant.)
- Spinors
- Quantum Electrodynamics (QED)

- Renormalization (dim. reg.; only QED and only 1-loop)
- Non-Abelian gauge theories, Higgs mechanism, "Standard Model"
- Non-relativistic QFT

- Path integral or functional integral (bosons, fermions, Feynman rules)
- Path integral quantization of non-Abelian gauge theories
- BRST symm symm. & phys. Hilbert space
- More on renormalization (renormalization group, running coupling & β-fct. in Quantum Chromodynamics or QCD)
- Infrared problems in QED & QCD; partons and operator product expansion in QCD
- Low-energy effective field theory, confinement, mesons, hadrons
- Non-perturbative effects (instantons, sphalerons)
- Anomalies
- QFT at $T \neq 0$; Conformal field theory or CFT

1.3 Literature

- Peskin / Schröder (our main text)
- Srednicki (a comparable comprehensive book)
- Itzykson/Zuber ; Weinberg I + II ; Bogoliubov/Shirkov (very detailed classical texts)
- Ryder ; Ramond (nice smaller books)
- Nachtmann ; Cheng/Li ; Donoghue/Golowich/Holstein ; Burgess/Moore $\rightarrow$ particle physics & SM
- Pokorski $\rightarrow$ gauge theories
- Collins $\rightarrow$ renormalization
- Rivers $\rightarrow$ path integral methods
- Montvay / Münster $\rightarrow$ lattice

2 Free scalar field

2.1 Classical theory

$A_\mu(x) \longrightarrow \varphi(x)$ (as a simple model, also relevant for pions & for Higgs)

$\mathcal{L}(\varphi, \partial_\mu\varphi) = \;?$ Main requirement: Lorentz-invariance

A) Derivative terms: Want non-trivial dynamics $\Rightarrow$ need $\dot{\varphi}$ $\Rightarrow$ need $\partial_\mu\varphi$

- $\partial_\mu\varphi \rightarrow \Lambda_\mu{}^\nu\partial_\nu\varphi \Rightarrow$ not Lorentz-inv. $\Rightarrow$ inacceptable
- $(\partial_\mu\varphi)(\partial^\mu\varphi)$ – OK
- $(\partial_\mu\varphi)(\partial^\mu\varphi)\cdot\varphi$; $(\partial_\mu\varphi\,\partial^\mu\varphi)^2$ etc. – less important

B) Non-derivative / potential terms:

- Any real fct. $V(\varphi)$ is OK (As above, high powers of φ are less important. Nevertheless it is useful to keep $V(\varphi)$ generic for the moment – for reasons that will become clear later.)

- Restricting ourselves to the dominant term with derivatives, we have

$$\mathcal{L} = \underset{\substack{\uparrow \\ \text{convention}}}{\frac{1}{2}}(\partial_\mu\varphi)(\partial^\mu\varphi) - V(\varphi)$$

(Note: The sign is not a convention since

$$(\partial_\mu\varphi)(\partial^\mu\varphi) = \dot{\varphi}^2 - (\nabla\varphi)^2 \quad , \quad \text{and} \quad L = \int d^3x\, \dot{\varphi}^2 + \cdots$$

is the natural generalization of $L = \sum_i \dot{q}_i^2 + \cdots$ of class. mechanics.)

- $V(\varphi) = c_0 + c_1\varphi + c_2\varphi^2 + \cdots$

c_0: irrelevant (at the moment)

$c_1\varphi$: can always be eliminated by $\varphi \rightarrow \varphi + \text{const.}$ if $c_2 \neq 0$.

<u>Free theory:</u> $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \underset{\substack{\uparrow \\ \text{convention.}}}{\frac{m^2}{2}}\varphi^2$ for now

$$L = \int d^3x\, \mathcal{L} = \int d^3\bar{x} \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\bar{\nabla}\varphi)^2 - \frac{m^2}{2}\varphi^2 \right)$$

$$(\varphi = \varphi(x) = \varphi(t, \bar{x}))$$

Thinking of a lattice with spacing Δ:

$$L = \sum_{\bar{x}} \left\{ \underbrace{\frac{1}{2}\dot{\varphi}(t,\bar{x})^2}_{T} - \underbrace{\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\varphi(t, \bar{x} + \hat{e}_i \Delta) - \varphi(t, \bar{x})}{\Delta} \right)^2 - \frac{m^2}{2}\varphi^2}_{\text{"}V\text{"} \;\; (V \text{ of QM})} \right\}$$

$\Rightarrow$ set of coupled harmonic oscillators (which we will decouple when we quantize them)

- Equation of motion:

$$\begin{aligned} 0 = \delta S &= \delta \int d^4x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 \right) \\ &= \int d^4x \left((\partial_\mu \varphi)(\partial_\nu \delta\varphi)\eta^{\mu\nu} - m^2 \varphi\, \delta\varphi \right) \\ &= -\int d^4x \left(\eta^{\mu\nu}\partial_\mu \partial_\nu \varphi + m^2\varphi \right) \delta\varphi \\ &= -\int d^4x \left((\partial^2 + m^2)\varphi \right) \delta\varphi \quad \Rightarrow \quad \underline{\underline{(\partial^2 + m^2)\varphi = 0}} \end{aligned}$$

Do this step carefully!

Klein-Gordon-equation

- solved e.g. by $\varphi(x) = \varphi_0 \sin kx$ with $\underline{\underline{k^2 - m^2 = 0}}$

(plane wave)

- $k = (k^0, \bar{k})$; $\bar{k} = 0 \Rightarrow k^0 = m$ ($\hat{=}$ particle with mass m)

Hamiltonian description:

- recall class. mechanics:

$$L(q_i, \dot{q}_i) \Rightarrow H(q_i, \pi_i) \text{ with } \pi_i = \frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i}$$

- field theory: $L[\varphi, \dot{\varphi}] = \int d^3x \left(\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right) =$

$$= \int d^3x \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\bar{\nabla}\varphi)^2 - \frac{m^2}{2}\varphi\right)$$

(We deliberately wrote $\varphi, \dot{\varphi}$ rather than φ, $\partial_\mu\varphi$ to be closer to mechanics.)

- $$\pi(\bar{x}) = \frac{\partial L}{\partial \dot{\varphi}(\bar{x})} = \frac{\partial}{\partial \dot{\varphi}(\bar{x})} \left\{ \text{"}\sum_{\bar{x}'}\text{"} \frac{1}{2}\dot{\varphi}(\bar{x}')^2 + \cdots \right\}$$

(t-dependence suppressed)

$$\Rightarrow \pi(\bar{x}) = \dot{\varphi}(\bar{x})$$

better formulation: $L = L[\varphi, \dot{\varphi}]$ - functional

$$\pi(\bar{x}) = \frac{\delta L}{\delta \dot{\varphi}(\bar{x})}$$

Let $F[f]$ be a functional; $\frac{\delta F}{\delta f(x)}$ is defined by

$$F[f+\varepsilon] - F[f] = \int dx \frac{\delta F}{\delta f(x)} \cdot \varepsilon(x) + O(\varepsilon^2)$$

$$\Rightarrow \frac{\delta L}{\delta \dot{\varphi}(\bar{x})} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\bar{x})} \quad \text{if } L = \int d^3x\, \mathcal{L}(\varphi, \dot{\varphi})$$

$$H[\varphi, \pi] = \int d^3x\, \pi\dot{\varphi} - L = \int d^3x \underbrace{\frac{1}{2}\left(\pi^2 + \bar{\nabla}\varphi^2 + m^2\varphi^2\right)}_{\text{"}\mathcal{H}\text{"}}$$

2.2 Quantization

- The usual postulate: $[\varphi(\bar{x}), \pi(\bar{y})] = i\delta^3(\bar{x}-\bar{y})$ $\quad (\hbar = c = 1)$

$$[\varphi(\bar{x}), \varphi(\bar{y})] = 0$$

$$[\pi(\bar{x}), \pi(\bar{y})] = 0$$

- We need to construct a Hilbert space representation of this operator algebra.

- Since $H = \int d^3x \, \frac{1}{2} \left(\pi^2 + (\bar{\nabla}\varphi)^2 + m^2\varphi^2 \right)$

  is very similar to a set of harmonic oscillators labelled by $\bar{x}$, it is tempting to change variables from

$$\pi, \varphi \quad \text{to} \quad a, a^+$$

  and continue in the well-known fashion.

- However, $(\bar{\nabla}\varphi)^2$ is non-diagonal, i.e. it couples the different harmonic oscillators. Thus, it is convenient to <u>decouple</u> them before introducing a, a^+.

- This is easily achieved by a Fourier transform:

$$\varphi(\bar{x}) = \int \frac{d^3p}{(2\pi)^3} \, e^{i\bar{p}\bar{x}} \, \tilde{\varphi}(\bar{p}) \quad ; \quad \tilde{\varphi}(\bar{p}) = \int d^3\bar{x} \, e^{-i\bar{p}\bar{x}} \, \varphi(\bar{x})$$

  and analogously for π, $\tilde{\pi}$.

- Let us derive the new commutation relations:

$$[\tilde{\varphi}(\bar{p}), \tilde{\pi}(\bar{q})] = \int d^3x \, d^3y \, e^{-i(\bar{p}\bar{x} + \bar{q}\bar{y})} [\varphi(\bar{x}), \pi(\bar{y})]$$

$$= i\int d^3x \, e^{-i(\bar{p}+\bar{q})\bar{x}} = i(2\pi)^3 \, \delta^3(\bar{p}+\bar{q})$$

this is like $\delta_{\bar{p},-\bar{q}}$, i.e., $\tilde{\pi}(\bar{p})$ is conjugate to $\tilde{\varphi}(-\bar{p})$ [This is ok since $\bar{p}, \bar{q}$ are just labels anyway.]

Also: $[\tilde{\varphi}(\bar{p}), \tilde{\varphi}(\bar{q})] = 0 \;;\; [\tilde{\pi}(\bar{p}), \tilde{\pi}(\bar{q})] = 0$

(obvious by linearity)

- Furthermore, we have to check that we have indeed successfully decoupled our oscillators:

$$\int d^3x \, (\bar{\nabla}\varphi)^2 = \int d^3x \int \frac{d^3p}{(2\pi)^3} \, (i\bar{p}) \, e^{i\bar{p}\bar{x}} \, \tilde{\varphi}(\bar{p}) \cdot$$
$$\cdot \int \frac{d^3q}{(2\pi)^3} \, (i\bar{q}) \, e^{i\bar{q}\bar{x}} \, \tilde{\varphi}(\bar{q})$$

Perform x-integration $\Rightarrow (2\pi)^3 \, \delta^3(\bar{p}+\bar{q}) \Rightarrow \bar{q} = -\bar{p}$

$$\cdots = \int \frac{d^3p}{(2\pi)^3} \, \bar{p}^2 \, \tilde{\varphi}(\bar{p}) \, \tilde{\varphi}(-\bar{p}) \, .$$

Observe that the reality of φ translates to $\varphi(x) = \varphi^+(x)$ at the operator level (reality of eigenvalues!), which translates to

$$\tilde{\varphi}^+(\bar{p}) = \tilde{\varphi}(-\bar{p}) \, .$$

$$\Rightarrow \quad \cdots = \int \frac{d^3p}{(2\pi)^3} \, \bar{p}^2 \, |\tilde{\varphi}(p)|^2 \leftarrow$$ sloppy notation for $\tilde{\varphi}\tilde{\varphi}^+$

The other terms in H are treated analogously, giving

$$H = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2} \left(|\tilde{\pi}|^2 + \omega_{\bar{p}}^2 \, |\tilde{\varphi}|^2 \right)$$

with $\omega_{\bar{p}} \equiv \sqrt{\bar{p}^2 + m^2}$ (in analogy to frequency of classical wave discussed above)

- Now everything looks extremely similar to the harmonic oscillator case, and it is time to recall the relevant formulae:

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) \quad ; \quad [q,p] = i$$

$$a = \frac{1}{2}\left(\sqrt{2\omega}\, q + i\sqrt{\frac{2}{\omega}}\, p\right) \; ; \; a^+ = \frac{1}{2}\left(\sqrt{2\omega}\, q - i\sqrt{\frac{2}{\omega}}\, p\right)$$

$$\downarrow$$

$$H = \omega\left(a^+ a + \frac{1}{2}\right) \quad ; \quad [a, a^+] = 1$$

- By analogy to harm. oscillator, define

$$a_{\bar p} = \frac{1}{2}\left(\tilde\varphi(\bar p)\sqrt{2\omega_{\bar p}} + \tilde\pi(\bar p)\, i\sqrt{\frac{2}{\omega_{\bar p}}}\right)$$

$$a_{\bar p}^+ = \frac{1}{2}\left(\tilde\varphi(-\bar p)\sqrt{2\omega_{\bar p}} - \tilde\pi(-\bar p)\, i\sqrt{\frac{2}{\omega_{\bar p}}}\right)$$

(Note the "–" in front of $\bar p$!)

or, equivalently,

$$\tilde\varphi(\bar p) = \frac{1}{\sqrt{2\omega_{\bar p}}}\left(a_{\bar p} + a_{-\bar p}^+\right), \quad \tilde\pi(\bar p) = -i\sqrt{\frac{\omega_{\bar p}}{2}}\left(a_{\bar p} - a_{-\bar p}^+\right)$$

- We now derive the commutation relations of $a_{\bar p}$, $a_{\bar p}^+$:

$$[a_{\bar p}, a_{\bar q}^+] = \frac{1}{2}\cdot\frac{1}{2}\cdot 2i\left(\sqrt{\frac{\omega_{\bar q}}{\omega_{\bar p}}}\,[\tilde\pi(\bar p), \tilde\varphi(-\bar q)] - \sqrt{\frac{\omega_{\bar p}}{\omega_{\bar q}}}\,[\tilde\varphi(\bar p), \tilde\pi(-\bar q)]\right)$$

$$= \frac{i}{2}\left((-i)(2\pi)^3\delta^3(\bar p - \bar q) - i(2\pi)^3\delta^3(\bar p - \bar q)\right)$$

not for blackboard

$$= (2\pi)^3\delta^3(\bar p - \bar q)$$

- Analogously, we find $[a_{\bar p}, a_{\bar q}] = 0$; $[a^+_{\bar p}, a^+_{\bar q}] = 0$.
- We also need to express H through a_p, a_p^+ :

$$H = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2} \left\{ \frac{\omega_{\bar p}}{2} (a_{\bar p} - a^+_{-\bar p})(a^+_{\bar p} - a_{-\bar p}) + \omega_{\bar p}^2 \cdot \frac{1}{2\omega_{\bar p}} \cdot (a_{\bar p} + a^+_{-\bar p})(a^+_{\bar p} + a_{-\bar p}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \cdot \frac{\omega_{\bar p}}{4} \left(a_{\bar p} a^+_{\bar p} + a^+_{\bar p} a_{\bar p} + a_{\bar p} a^+_{\bar p} + a^+_{\bar p} a_{\bar p} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \; \omega_{\bar p} \left(a^+_{\bar p} a_{\bar p} + \frac{1}{2} \underbrace{[a_{\bar p}, a^+_{\bar p}]}_{} \right)$$

$$(2\pi)^3 \delta^3(\bar 0) \text{ "="} \int d^3\bar x \, e^{i \bar 0 \cdot \bar x} = Vol.(\mathbb{R}^3) = V$$

$$H = \int \frac{d^3\bar p}{(2\pi)^3} \; \omega_p \, a^+_{\bar p} a_{\bar p} \; + V \int \frac{d^3\bar p}{(2\pi)^3} \cdot \frac{1}{2} \omega_{\bar p}$$

The argument leading to this factor could be made "clean" by considering T^3 instead of $\mathbb{R}^3$.

This divergence is "for real": In QFT the vacuum energy <u>density</u> is still divergent.

We will ignore this infinite constant for the moment.
It is, however, important:

- When QFT is coupled to gravity, it leads to the so-called "cosmological constant problem"
- Its finite change by the motion of boundaries (conducting plates in QED) leads to the "Casimir effect").

Summary: $H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\bar p}\, a^+_{\bar p} a_{\bar p}$, $\omega_{\bar p} = \sqrt{m^2 + \bar p^2}$

$[a_p, a^+_{\bar q}] = (2\pi)^3 \delta^3(\bar p - \bar q)$, $\varphi(\bar x) = \cdots$
$\pi(\bar x) = \cdots$ see above

$\Rightarrow$ We have found a continuum of harmonic oscillators labelled by $\bar p$ with frequencies (i.e. energies) $\omega_{\bar p}$.

- We will interpret the excitations of this system as particles with momenta $\bar p$ (to be justified in more detail later).
- At the moment, we accept this "particle" notion as a convenient language and construct our Hilbert space in complete analogy to a set of harmonic oscillators:
- Vacuum: $|0\rangle$: Defined as a state with $a_{\bar p}|0\rangle = 0$ (for all $\bar p$)
- One-particle states: $a^+_{\bar p}|0\rangle$ (for any $\bar p$)

$$H\left(a^+_{\bar p}|0\rangle\right) = \int \frac{d^3q}{(2\pi)^3}\, \omega_{\bar q}\, a^+_{\bar q} a_{\bar q}\, a^+_{\bar p}|0\rangle$$

$$= \int d^3q\, \omega_{\bar q}\, a^+_{\bar q}\, \delta^3(\bar p - \bar q)|0\rangle = \omega_{\bar p}\left(a^+_{\bar p}|0\rangle\right),$$

in agreement with our suggested interpretation. (The energy of a relativistic particle with mass m and momentum $\bar p$ is $\sqrt{m^2 + \bar p^2}$.)

- Two-particle states: $a^+_{\bar p} a^+_{\bar q}|0\rangle$ (for any $\bar p$ & $\bar q$).

$$H\left(a^+_{\bar p} a^+_{\bar q}|0\rangle\right) = (\omega_{\bar p} + \omega_{\bar q})\left(a^+_{\bar p} a^+_{\bar q}|0\rangle\right)$$

(Check this!)

- More particles: analogously
- This space is also known as "Fock space".
- Normalization: $||0\rangle|^2 = \langle 0|0\rangle = 1$ by definition

$$\Rightarrow \left(\langle 0| a_{\bar p}\right)\cdot\left(a_{\bar q}^{+}|0\rangle\right) = \langle 0| a_{\bar p} a_{\bar q}^{+} |0\rangle = (2\pi)^3 \delta^3(\bar p - \bar q)$$

- We will use 1-particle states in the notation and normalization

$$|\bar p\rangle \equiv \sqrt{2\omega_{\bar p}}\, a_{\bar p}^{+}|0\rangle ,$$

such that $$\langle \bar p | \bar q\rangle = 2\omega_{\bar p} (2\pi)^3 \delta^3(\bar p - \bar q).$$

This is convenient since, integrating over all possible momenta of a particle of mass m in a Lorentz-invariant way, one finds a Lorentz-invariant result:

– 4-momentum: $p = (p^0, \bar p) = (\omega_{\bar p}, \bar p)$

$$\underbrace{\int d^4p\, \delta(p^2 - m^2)\Big|_{p^0>0}}_{\text{manifestly Lorentz-invar.}} 2\omega_{\bar p} (2\pi)^3 \delta^3(\bar p - \bar q) = (2\pi)^3$$

$$= \int \frac{d^3p}{2\omega_{\bar p}} \qquad \text{(Check this!)}$$

- <u>Note:</u> Frequently, a different normalization for creation/annihilation operators is used: $a'_{\bar p} = \sqrt{2\omega_{\bar p}}\, a_{\bar p}$.

Advantage: $|p\rangle = a_{\bar p}'^{+}|0\rangle$

Disadvantage: $[a'_{\bar p}, a_{\bar q}'^{+}] = 2\omega_{\bar p} (2\pi)^3 \delta^3(\bar p - \bar q)$

(Cf. Peskin/Schröder vs. Srednicki)

different from usual harm. oscillator

2.3 Complex scalar field (just results, for proofs → tutorials)

$$\mathcal{L} = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - m^2 \phi \phi^* = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

(commonly used sloppy notation)

Can be reduced to real case by $\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$

$\Rightarrow \phi\phi^* = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$ and analogously for $\partial_\mu \phi \partial^\mu \phi^*$

However, it is nevertheless useful to redo all we have done in terms of ϕ & ϕ^*

Treat ϕ & ϕ^* formally as indep. quantities

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -m^2 \phi^* - \underset{\substack{\uparrow \\ \partial_\mu \partial^\mu}}{\Box} \phi^*$$

analogously, or simply by compl. conjugation, we also have

$$(\Box + m^2)\phi = 0 \quad \text{(K.-G.-Eq.)}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi^* \; ; \quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot\phi^*} = \dot\phi$$

$$\mathcal{H} = \pi \dot\phi + \pi^* \dot\phi^* - \mathcal{L} = |\pi|^2 + |\bar\nabla \phi|^2 + m^2 |\phi|^2$$

Quantization: $\quad [\phi(\bar x), \pi(\bar y)] = [\phi^+(\bar x), \pi^+(\bar y)] = i\delta^3(\bar x - \bar y)$

Ansatz for ϕ: $\quad \phi(\bar x) = \displaystyle\int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\bar p}}} \left(a_{\bar p}^+ e^{-i\bar p \bar x} + b_{\bar p} e^{i\bar p \bar x} \right)$

Note that we need a second set of creation / annihilation operators

$\Rightarrow$ Two types of particles, created by $a_{\bar p}^+$ & $b_{\bar p}^+$.

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\bar{p}} \left(a^+_{\bar{p}} a_{\bar{p}} + b^+_{\bar{p}} b_{\bar{p}} \right)$$

The symmetry $\phi \to e^{i\alpha}\phi$ of $\mathcal{L}$ leads to a conserved quantity, Q (cf. Noether theorem of class. mechanics), which reads

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(\underset{\substack{\uparrow \\ \text{particles with} \\ \text{charge } +1}}{a^+_{\bar{p}} a_{\bar{p}}} - \underset{\substack{\uparrow \\ \text{antiparticles} \\ \text{with charge } -1.}}{b^+_{\bar{p}} b_{\bar{p}}} \right)$$

To derive this result, we will have to discuss the Noether theorem in the context of QFT.

<u>Note:</u> If we "gauge" this symm., i.e. allow for trfs. $\phi(x) \to e^{i\alpha(x)}\phi(x)$, we will be forced to couple ϕ to a field $A_\mu(x)$, and the above charge will really be the familiar electromagnetic charge.
$\to$ later in this lecture