10 Quantum-Electrodynamics

10.1 Lagrangian

As in scalar case before, start with the free fermionic lagrangian and "gauge" its $U(1)$-symmetry:

$$\mathcal{L}_\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

- global symm: $\psi \to e^{-i\alpha}\psi$ (with α = const.)
- desired local symm: $\psi \to e^{-i\alpha(x)}\psi$
- replace $\partial_\mu \longrightarrow D_\mu = \partial_\mu + iA_\mu$
- symm. tf. of $D_\mu\psi$: $D_\mu\psi \longrightarrow D'_\mu\psi' = (\partial_\mu + iA'_\mu)e^{-i\alpha(x)}\psi$

$$= e^{-i\alpha}\partial_\mu\psi - i(\partial_\mu\alpha)e^{-i\alpha}\psi + iA'_\mu\psi$$

$$= e^{-i\alpha}D_\mu\psi$$

$$\text{if } A'_\mu = A_\mu + \partial_\mu\alpha .$$

- Since $\bar{\psi} \to e^{i\alpha}\bar{\psi}$, this makes the lagrangian

$$\mathcal{L} = \bar{\psi}(iD\!\!\!\!/ - m)\psi \quad \underline{\text{gauge-invariant.}}$$

- Adding a kinetic term for our new field A_μ (which also has to be gauge-invariant!), we find:

$$\mathcal{L}_{QED} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!\!/ - m)\psi$$

or

$$\mathcal{L}_{QED} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{canonical kinetic term for gauge field.}} + \bar{\psi}(iD\!\!\!\!/ - m)\psi \quad \text{with } D_\mu = \partial_\mu + ieA_\mu$$

10.2 Feynman rules

Split $\mathcal{L}_{QED}$ in free part and interaction lagrangian:

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int.}$$

$$\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi \quad ; \quad \mathcal{L}_{int.} = -e\bar{\psi}\not{A}\psi$$

This way of coupling A to ψ as required by gauge-inv. is called minimal coupling.

(A non-minimal coupling would be, e.g. $\frac{1}{\Lambda} F_{\mu\nu} \bar{\psi}\gamma^\mu\gamma^\nu\psi$.

Including such a term in the fund. Lagrangian would make the theory non-renormalizable (see later). If Λ is sufficiently large, it is also irrelevant since it leads to corrections of the type (at low energy)

$$1 + O(1)\cdot\frac{\sqrt{s}}{\Lambda}.$$

It is called a "higher-dim." operator because its mass dimension is 5 (hence Λ has mass-dim. 1).

$\mathcal{L}_{free}$ leads to

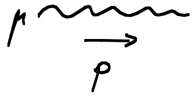
$$a \xrightarrow[p]{} b = \left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{ba}$$

(What we have actually derived is $\frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$. However, since ε is only responsible for circumnavigating the poles, this is equivalent:

$$\frac{i}{\not{p} - m + i\varepsilon} = \frac{\not{p} + (m - i\varepsilon)}{(\not{p} + (m - i\varepsilon))(\not{p} - (m - i\varepsilon))} = \dots$$

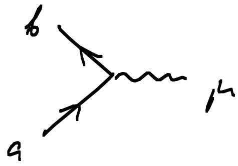
$$\cdots \approx \frac{\not{p}+m}{p^2-(m-i\varepsilon)^2} \approx \frac{\not{p}-m}{p^2-m^2+i\varepsilon}\,.\Big)$$

and to



$$= \frac{-i\eta^{\mu\nu}}{p^2+i\varepsilon} \qquad \text{(in Feynman gauge).}$$

$\mathcal{L}_{int.}$ obviously leads to



$$= e\cdot(\gamma_\mu)_{ba}\cdot \text{some constant}$$

($e\bar{\Psi}A^\mu\gamma_\mu\Psi$, with $\Psi, \bar{\Psi}, A^\mu$ used for contraction fermionic / bosonic fields in the rest of the diagram.)

This we will derive (and determine the constant using, as before, an "imagined" process $e^+ + \gamma \to e^+$:

(momenta: $p + k = p'$)

$$\langle 0|\, a^{s'}_{\bar{p}'} \left(i\int d^4x\, \mathcal{L}_{int.}\right) a^{s+}_{\bar{p}}\, \underbrace{a^{\mu+}_{\bar{k}}\, \varepsilon_\mu(k)}\, |0\rangle$$

outgoing e^+ with spin s' — incoming e^+ with spin s — incoming γ with polarization vector ε_μ

We have

$$\langle 0|\, a^{s'}_{\bar p'} \left[(-ie) \int d^4x\, \bar\psi(x)\, \gamma_\nu A^\nu(x)\, \psi(x) \right] a^{s+}_{\bar p}\; a^{\mu +}_{\bar k} |0\rangle\, \varepsilon_\mu(k)$$

Recall:

$$\psi(x) = \int d\tilde q\; a^r_{\bar q}\, u_r(q)\, e^{-iqx} + \cdots$$

$$\bar\psi(x) = \int d\tilde q\; a^{r+}_{\bar q}\, \bar u_r(q)\, e^{iqx} + \cdots$$

$$A^\nu(x) = \int d\tilde q\; (a_{\bar q})^\nu\, e^{-iqx} + \cdots$$

and apply:

$$\{ a^r_{\bar q}, a^{s+}_{\bar p} \} = 2p_0 (2\pi)^3 \delta^3(\bar q - \bar p)\, \delta^{rs}$$

$$[(a_{\bar q})^\nu, (a_{\bar k})^{+\mu}] = -\, 2k_0\, (2\pi)^3\, \delta^3(\bar q - \bar k)\, \eta^{\nu\mu}$$

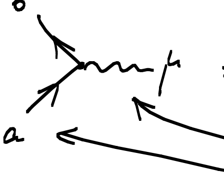
recall that the sign should be "right" for the space like polarizations and that our metric is $(+,-,-,-)$.

After a small orgy integration (which however can be done by "carefully looking at the expression") we find:

$$\bar u_{s'}(p')\, (ie\gamma^\mu)\, u_s(p)\, \varepsilon_\mu(k).$$

(To be multiplied by $\varepsilon_\mu(k)$, if you want, which we already know to correspond to an incoming photon.)

We conclude:

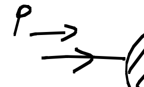


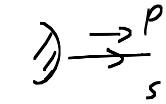
$$= ie(\gamma^\mu)_{ba}\,,$$

(as also above)

together with:

amputated!

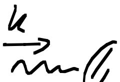
 $= u_s(p)_a$ (incoming positron)

 $= \bar{u}_s(p)_a$ (outgoing positron)

This index is usually suppressed since matrix notation already suggests how this is to be contracted with a vertex:

$$\bar{u}(ie\gamma^\mu)u.$$
$$\uparrow \qquad \uparrow$$
$$b\ b \qquad a\ a$$

As before:

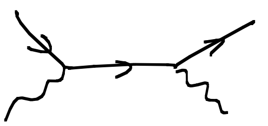
 $= \varepsilon_\mu(k)$ (incoming photon, with "$*$" for outgoing).

Note: As for the complex scalar, the propagator is directed (has an arrow). This is not only important for writing the correct vertex (see scalar case), but also because S_F is not invariant under $x \leftrightarrow y$. To get some intuition for these issues, consider a simple example:

$e^+\gamma \to e^+\gamma$, i.e. (very brief notation)

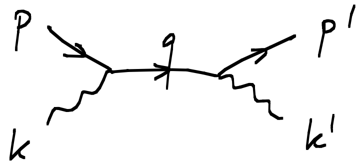
$$\langle 0|\, a^s a^r \,\Big|\Big(\int_x \bar{\psi}\,(-ie\gamma_\nu A^\nu)\,\psi\Big)\Big(\int_y \bar{\psi}\,(-ie\gamma_\beta A^\beta)\,\psi\Big)\,\Big|\, a^{s+} a^{r+}|0\rangle$$

with the corresponding diagram

incoming {  } outgoing

This shows that our conventions are such that the arrow on the propagator goes with the momentum of the positron (particle! charge +) (same for the incom./outgoing lines).

It also shows how to use efficiently the matrix notation for the propagator:



$$= \varepsilon_\mu(k')\,\bar{u}_{s'}(p')(ie\gamma^\mu)\frac{i}{\not{q}-m+i\varepsilon}(ie\gamma^\nu)\,u_s(p)\,\varepsilon_\nu(k)$$

to be read from left to right

to be read from right to left, contracting spinor indices as one goes along the propagator (following the arrow).

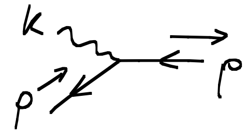
- For the electron (antiparticle), things are a bit different:

Consider, e.g., "$e^-\gamma \to e^-$" with $p+k=p'$:

$$\text{"}\langle 0|\, b_{s'} \left(\int \bar\psi\,(-ie\not{A})\,\psi\right) b_s^+\, a^{\mu+}|0\rangle$$

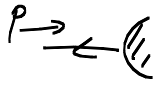
analogous calculation (taking b^+ from ψ and b from ψ^+)

$$\underbrace{\bar{v}_s(p)\,(ie\gamma^\mu)\,v_{s'}(p')}_{\text{"left to right"}} =$$

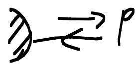


"left to right"

giving the Feynman rules:



$= \bar{v}_s(p)_a$ — incoming electron

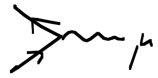


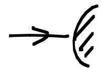
$= v_s(p)_a$ — outgoing electron

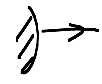
<u>Summary</u>

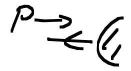
 $= i/(\not k - m + i\varepsilon)$

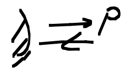
 $= -i g^{\mu\nu}/(k^2 + i\varepsilon)$

 $= ie\gamma^\mu$

 $= (\,\cdots)\cdot u$ incoming part.

 $= \bar u(\cdots)$ outgoing part.

 $= \bar v(\cdots)$ incom. antipart.

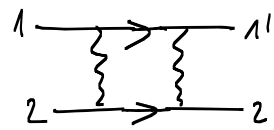
 $= (\cdots)\, v$

some expression with γ's.

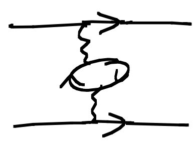
incom./outg. photon: ε^μ / $\varepsilon^{\mu *}$

Extra: A (relative) minus sign for every closed fermion loop <u>and</u> for exchanging two external fermion-momenta between two diagrams

- We demonstrate the first case with an example: two diagrams contributing to $e^+e^+ \to e^+e^+$ at order e^4 (for the amplitude):

① 

$$\langle 2'1'(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)1\,2\rangle$$

② 

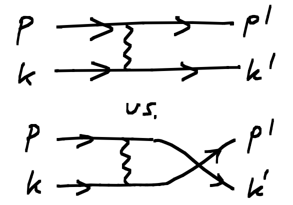
$$\langle 2'1'(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)(\bar\psi A\psi)1\,2\rangle$$

- each side acquires 2 two minus-signs associated with "intersecting" fermionic contraction lines (see above)
- in addition, on the r.h. side we have one contraction

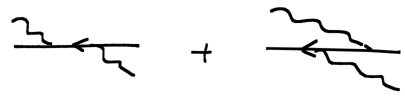
$\bar\psi\psi$ (rather than $\psi\bar\psi$, as all the others).

$-\psi\bar\psi \Rightarrow$ This is the extra minus coming from the loop!

2nd case:

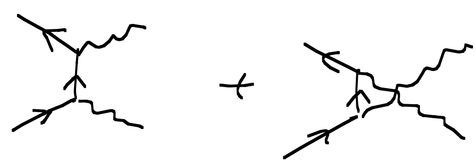


10.3 Elementary processes

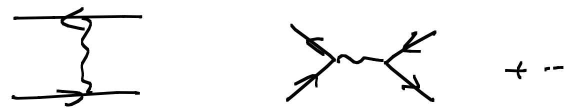
- Compton scattering : $e^-\gamma \to e^-\gamma$; 

  ($\to$ Klein/Nishina formula)

- elastic e^-e^--scattering (or e^+e^+) – see above
  - one of them non-relativistic (e.g. μ or nucleus)
    $\to$ Coulomb scattering (Mott formula)
  - both non-relativistic
    $\to$ Rutherford scatter. (Rutherford formula)

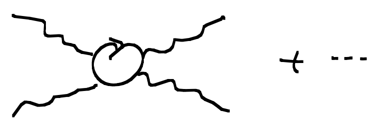
- pair-annihilation to photons : 

  $e^+e^- \to \gamma\gamma$

- Bhaba-scattering : 

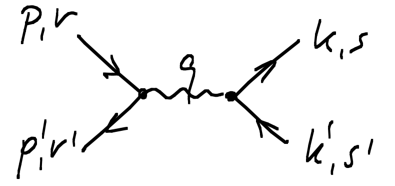
  $e^+e^- \to e^+e^-$

- Light-by-light scattering (only at loop level)



Our example calculation:

$e^+e^- \to \mu^+\mu^-$ (particularly simple since only one diagram contributes ; also practically important)



incoming outgoing

$(q^2 = (p+p')^2 = (k+k')^2 = s\ )$

$$i\mathcal{M} = \bar{u}_s(k)(ie\gamma_\mu)v_{s'}(k')\left(\frac{-i\eta^{\mu\nu}}{q^2}\right)\bar{v}_{r'}(p')(ie\gamma_\nu)u_r(p)$$

For the cross section

$$d\sigma = \int \frac{1}{2s} |\mathcal{M}|^2 dx^{(2)} = \frac{1}{64\pi^2 s} |\mathcal{M}|^2 d\Omega$$

at $\sqrt{s} \gg m_e, m_\mu$ we need (if we don't polarize/measure spin):

$$|\mathcal{M}|^2 \to \underbrace{\frac{1}{2}\sum_r \frac{1}{2}\sum_{r'}}_{\text{average}} \underbrace{\sum_s \sum_{s'}}_{\text{sum}} |\mathcal{M}(r,r',s,s')|^2 = \cdots$$

$$\cdots = \frac{e^4}{4s^2} \underbrace{\sum_{s,s'} \left(\bar{u}_s(k)\gamma_\mu v_{s'}(k')\right)\left(\bar{u}_s(k)\gamma_\nu v_{s'}(k')\right)^*}_{\equiv A_{\mu\nu}} \underbrace{\sum_{r,r'} \left(\bar{v}_{r'}(p')\gamma^\mu u_r(p)\right)\left(-\text{''}-\right)^*}_{\equiv B^{\mu\nu}}$$

$$A_{\mu\nu} = \sum_{s,s'} tr\left[u_s(k)\bar{u}_s(k)\gamma_\mu v_{s'}(k')\bar{v}_{s'}(k')\gamma_\nu\right]$$

(think of spinors as 1×4 or 4×1 matrices!)

$$= tr\left[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right]$$

$$= tr\left[\not{k}\gamma_\mu\not{k}'\gamma_\nu\right] - m_\mu^2\, tr\left[\gamma_\mu\gamma_\nu\right]$$

since $tr(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0$ for n odd. *

* Proof: $tr(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = tr(\gamma^5\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n})$

$$= -tr(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5) = -tr(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) \Rightarrow = 0$$

- $tr[\gamma_\mu\gamma_\nu] = \frac{1}{2} tr[\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu] = \eta_{\mu\nu}\, tr(\mathbb{1}) = 4\eta_{\mu\nu}$
- $tr[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = tr[2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma - \gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma]$

$$= tr[2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma] = \cdots$$

$$= 8\eta_{\mu\nu}\eta_{\rho\sigma} - 8\eta_{\mu\rho}\eta_{\nu\sigma} + 8\eta_{\nu\rho}\eta_{\mu\sigma} - tr[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma]$$

$$\Rightarrow \mathrm{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma})$$

altogether:

$$A_{\mu\nu} = 4(k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k') - \eta_{\mu\nu} m_\mu^2) \longrightarrow [\to 0 \text{ for } m_\mu^2 \ll s]$$

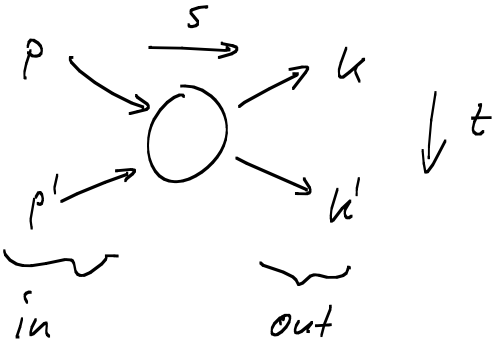
[$B_{\mu\nu}$ contains m_e^2 instead of m_μ^2, which we of course also neglect.]

$$(\text{sum \& aver.})\,|\mathcal{M}|^2 = \frac{e^4}{4s^2}\cdot 16 \cdot (k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(kk'))\underbrace{(p'^\mu p^\nu + \cdots)}_{\text{with } k\to p}$$

$$= \frac{8e^4}{s}((kp)(k'p') + (kp')(k'p)) = \cdots$$

Recall Mandelstam variables:

$$s = (p+p')^2$$
$$t = (p-k)^2$$
$$u = (p-k')^2$$

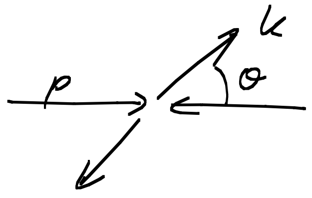


$$\left(s+t+u = \sum_{i=1}^{4} m_i^2\right)$$

here: $p^2 = \cdots = k'^2 = 0 \quad \Rightarrow \quad s = 2pp' = 2kk'$
$$t = -2kp = -2p'k'$$
$$u = -2pk' = -2p'k$$

$$\Rightarrow \cdots = 2e^4\,\frac{t^2+u^2}{s^2} = \cdots$$

- We now need to express this through the measured angle θ:



(in cms, where $\bar{p} = -\bar{p}'$, $\bar{k} = -\bar{k}'$)

$$t = -2pk = -2(p_0 k_0 - \vec{p}\vec{k}) = -2p_0 k_0 (1-\cos\theta)$$

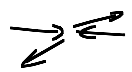
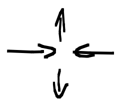
$$= -2\left(\frac{\sqrt{s}}{2}\right)^2 (1-\cos\theta) = -\frac{s}{2}(1-\cos\theta)$$

analogously: $u = -\frac{s}{2}(1+\cos\theta)$

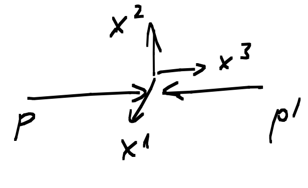
$$\Rightarrow \;\; --- = e^4 (1+\cos^2\theta)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}(\,\cdot\cdot\,) = \frac{e^4}{64\pi^2 s}(1+\cos^2\theta) = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

$$(d\Omega = d\varphi \sin\theta \, d\theta)$$

$\Rightarrow$  preferred w.r.t.  .

- Physical explanation of this effect:
  - The intermediate state can be viewed as a massive vector particles with 3 allowed polarizations ($\varepsilon \cdot q = 0$, $q = (\sqrt{s}, \vec{0})$).
  - Choose $\varepsilon^{(1)} = (0,1,0,0)$; $\varepsilon^{(2)} = (0,0,1,0)$, $\varepsilon^{(3)} = (0,0,0,1)$, and a frame where the beam axis coincides with the x^3- direction:



  - $\varepsilon^{(3)} \sim p - p' \Rightarrow \bar{v}(p')(-ie\gamma^\mu)u(p) \cdot \varepsilon^{(3)}_\mu = 0$
    since $\bar{v}(p')\not{p}' \approx 0$ & $\not{p}u(p) \approx 0$.
  - Hence only $\varepsilon^{(1)}, \varepsilon^{(2)}$ (equivalently: $\varepsilon^{(\pm)}$ along 3-axis) are produced. Intuitive reason: spin 1/2 + spin 1/2 = spin 1
  - By an analogous argument for the final state spinor expression $\bar{u}(k)\gamma^\mu v(k')$, the photon "likes" to decay along its polarization axis x^3. Hence the enhancement at $\vec{k}, \vec{k}' \parallel \hat{e}_3$.

A more careful argument (using the density matrix concept):

$$\sum_{\text{spins}} (\bar{v}\gamma^\mu u)(\bar{v}\gamma^\nu u)^* \longrightarrow \varrho_{\text{initial}} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\varepsilon^0 = 0$ by gauge choice

$$\sum_{\text{spins}} (\bar{u}\gamma^\mu v)(\bar{u}\gamma^\nu v)^* \longrightarrow \varrho_{\text{final}} \sim \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \bar{k} \parallel \bar{p} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \bar{k} \perp \bar{p} \end{cases}$$

This follows directly from our expressions for $A^{\mu\nu}$, $B^{\mu\nu}$ above.
It also follows, in a more intuitive way (cf. our discussion of "spin 1/2 + spin 1/2 = spin 1") from

$$\sum_{\pm} (\varepsilon^\mu_\pm)(\varepsilon^\nu_\pm)^* \to \sum_{\pm} (\varepsilon^i_\pm)(\varepsilon^i_\pm)^* = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \cdot (1 \; -i \; 0) + \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \cdot (1 \; i \; 0)$$

$$= \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \varrho_{\text{initial}}$$

(with ϱ_{final} simply following by conventional $SO(3)$-rotation for generic $\bar{k}$)

<u>This can be written as:</u>

$$\varrho_{\text{initial}} \sim \delta_{ij} - \hat{p}_i \hat{p}_j \qquad (\hat{p} \equiv \bar{p}/|\bar{p}|)$$

$$\varrho_{\text{final}} \sim \delta_{ij} - \hat{k}_i \hat{k}_j$$

$$\mathrm{tr}(\varrho_{\text{initial}}\, \varrho_{\text{final}}) \sim 3 - 1 - 1 + (\hat{k}\cdot\hat{p})^2 = 1 + \cos^2\theta$$