11 Renormalization

(very brief & only for QED)

11.1 Concept

- Higher order pert. calculations of many quantities (cross-sections, self-energies, Greens-fcts. etc.) will involve divergent loop integrals. For the moment, let's regularize them using a momentum-cutoff Λ.

- Furthermore, let's write our familiar Lagrangian as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu\,0} F_0^{\mu\nu} + \bar{\psi}_0 \left(i\left(\not{\partial} + i e_0 \not{A}_0\right) - m_0 \right)\psi$$

$$\left(F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu \right)$$

and call the familiar fields & couplings appearing in it "bare". (Hence they all got the index "0".)

- Next, let us rewrite it in new, renormalized variables (both for fields & couplings) according to

$$A_0^\mu = Z_A^{1/2} A^\mu \;;\quad \psi_0 = Z_\psi^{1/2}\psi \;;\quad e_0 = Z_e \cdot e \;;\quad m_0 = Z_m m$$

$$\mathcal{L} = -\frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu} + Z_\psi \left(i\left(\not{\partial} + i Z_e Z_A^{1/2} e \not{A}\right)\right) - Z_m m \Big)\psi$$

Note that this still describes the same physics.

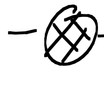
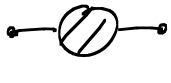
- Crucial idea: Let $Z_i = Z_i(\Lambda)$ and choose the functional dependence on Λ such that all relevant quantities (cross sects., self-energies, ...) approach a finite limit as $\Lambda \to \infty$.

- If this is possible, the theory is called renormalizable. [Note that this is highly non-trivial since there are only 3 Z's but infinitely many "relevant quantities".]

- By trading e_0 for $e, Z_e, Z_A, \cdots$, we have clearly introduced a redundancy. To fix this & calculate the Z_i explicitly, we impose renormalization conditions (e.g. fix poles & residues of all propagators & as many cross sections as we have independent couplings).

11.2 Renormalization conditions

① Reminder of scalar case:

 $= -i\Pi(p^2)$;  $= \dfrac{i}{p^2 - m^2 - \Pi(p^2)}$

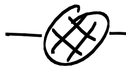
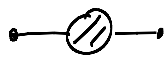
$\uparrow$ mass in $\mathcal{L}$ (we previously called it m_0, but now this is used for the bare mass. Our mass in $\mathcal{L}_{ren}$, with which we always work, is m)

mass: $m_{phys.}^2 - m^2 - \Pi(m^2) = 0 \Rightarrow m_{phys} = m$ means $\underline{\underline{\Pi(m^2) = 0}}$

Z: $Z^{-1} = 1 - \Pi'(m^2) \Rightarrow Z = 1$ means $\underline{\underline{\Pi'(m^2) = 0}}$

② electron:

can't write p^2, since terms $\sim p^\mu \gamma_\mu$ appear.

$_a$  $_b = -i\,\Sigma(\not{p})_{ab}$;  $= \dfrac{i}{\not{p} - m - \Sigma(\not{p})}$

$\uparrow$ 4x4-matrix!

<u>Claim:</u> Requiring $m = m_{phys.}$ & $Z = 1$ corresponds to

$$\Sigma(m) = 0 \quad \& \quad \Sigma'(m) = 0 .$$

<u>Demonstration:</u> In the Z-factor calculation for the scalar (cf. Sect. 6.6), we arrived at
$$\frac{i}{p^2 - m^2 - \Pi(p^2)} = \frac{iZ}{p^2 - m_{phys.}^2} + \cdots ,$$

and Z & $m_{phys.}$ followed by demanding that pole-position & residue agree. The analogous formula for spinors is (already assuming $m = m_{phys.}$ & $Z = 1$):

$$\frac{i}{\not p - m - \Sigma(\not p)} = \frac{i}{\not p - m} + \underbrace{\cdots}_{\text{no poles.}}$$

[Note that the conclusion $\Sigma(m) = \Sigma'(m) = 0$ is <u>not yet</u> obvious since we want the poles to agree at $p = p_{\text{on-shell}}$, not at $\not p = m$.]

- We write $\Sigma(\not p) = \Sigma(m) + \Sigma'(m)(\not p - m) + \frac{1}{2}\Sigma''(m)(\not p - m)^2 + \cdots$

  [Note that we <u>do not</u> require this to be a good approx. at any finite order since $\not p$ is <u>not</u> close to m.]

If we now impose $\Sigma(m) = \Sigma'(m) = 0$, our equation above becomes

$$\frac{i}{(\not p - m)\left(1 + \frac{1}{2}\Sigma''(m)(\not p - m) + \cdots\right)} = \frac{i}{\not p - m} + \cdots$$

$$\frac{i(\not p + m)}{(p^2 - m^2)\left(1 + \frac{1}{2}\Sigma''(m)(\not p - m) + \cdots\right)} = \frac{i(\not p + m)}{p^2 - m^2} + \cdots$$

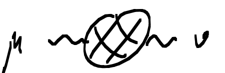
some other series in $(\not p - m)$.

$$\frac{i(\not p + m)\left(1 - \frac{1}{2}\Sigma''(m)(\not p - m) + \cdots\right)}{p^2 - m^2} = \frac{i(\not p + m)}{p^2 - m^2} + \cdots$$

$$\frac{i(\not p + m)}{p^2 - m^2} + i\left[\frac{1}{2}\Sigma''(m) + \cdots\right] = \frac{i(\not p + m)}{p^2 - m^2} + \cdots$$

obviously no poles at $p \to p_{\text{on-shell}}$.
hence $\Sigma = \Sigma' = 0$ at $\not p = m$ has indeed "done the job".

③ Photon:

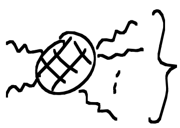
μ  ν $= i\Pi_{\mu\nu}(q^2)$;  $= \dfrac{i}{-q^2\eta_{\mu\nu} + \Pi_{\mu\nu}(q^2)}$

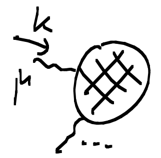
- We clearly need $\Pi_{\mu\nu}(0) = 0$ to ensure that photon remains massless (as required by gauge inv.)
- Most general form: $\Pi_{\mu\nu}(q^2) = \eta_{\mu\nu} A(q^2) + q_\mu q_\nu B(q^2)$
- gauge inv. fixes the ration of A & B:

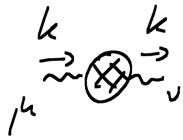
$$\Pi_{\mu\nu}(q^2) = (\eta_{\mu\nu} q^2 - q_\mu q_\nu)\,\Pi(q^2)$$

don't confuse with Π of scalar case!

Rough argument for this:

- Consider $2 \to n$ photon amplitude:  $\}\, n$
- It should not matter whethe we contract external line with momentum k with $\varepsilon^\mu(k)$ or $\varepsilon^\mu(k) + \alpha k^\mu$

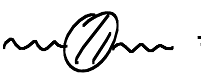
$\Rightarrow$  $\cdot\, k^\mu = 0$

- The lowest-order case of this is  $\cdot\, k^\nu = 0$

or

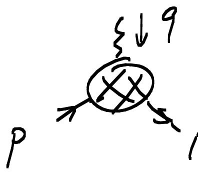
$\Pi_{\mu\nu}(q)\, q^\nu = 0$, hence the relation between A & B

- If we focus on phys. polarizations ($\varepsilon^\mu(q)\, q_\mu = 0$),

 $= \dfrac{i}{-\eta^{\mu\nu} q^2 + \eta^{\mu\nu} q^2 \Pi(q^2)} = \dfrac{-i\eta_{\mu\nu}}{q^2(1 - \Pi(q^2))}$

i.e., $Z = 1$ means $\Pi(0) = 0$ (This is analogous to $\Pi'(m^2) = 0$ in the scalar case because of the extra prefactor q^2.)

④ Vertex (instead of scattering cross section)



$$= ie\,\Gamma^\mu(p,p')$$

Demand that this is the same as what the tree-level Lagrangian with coupling e would predict at zero photon momentum:

$$\Rightarrow \quad \Gamma^\mu(p,p) = \gamma^\mu .$$

11.3 QED β-fct.

By definition, the renormalized coupling does not depend on Λ:

$$e_0(\Lambda) = Z_e(\Lambda)\cdot e \quad \Rightarrow \quad \frac{d}{d\ln\Lambda} e_0(\Lambda) = e\,\frac{d}{d\ln\Lambda} Z_e(\Lambda)$$

$$\overset{\text{at LO}}{\approx} e_0 \frac{\partial}{\partial \ln\Lambda}\left(1 + c\,e_0^2 \ln\Lambda\right)$$

$$= e_0^3 c$$

$\Rightarrow$ to get $\beta(e_0) \equiv \frac{d}{d\ln\Lambda} e_0(\Lambda) \approx e_0^3\cdot c$,

we just need the coeff. of the $e^2\ln\Lambda$-term in Z_e.

- There is an easy way of getting this by observing that gauge invariance requires the structure

  $\partial_\mu + ieA_\mu$ to be unchanged in renormalization,

  hence $Z_e Z_A^{1/2} = 1$ or $c = -\frac{1}{2}\cdot\frac{1}{e^2}\frac{d}{d\ln\Lambda} Z_A$.

Note: historically, people use

$$Z_A \equiv Z_3 \; ; \; Z_\psi \equiv Z_2 \quad \text{and} \quad Z_e Z_2 Z_3^{1/2} \equiv Z_1 .$$

The above (famous and important) relation then reads $Z_1 = Z_2$.

follows from "Ward identities"

to be discussed more carefully in part QFT II

- $Z_A = 1 + \delta Z_A$ induces the counterterm

$$\mu \sim\!\!\times\!\!\sim \nu \;=\; i\left(-\eta^{\mu\nu} p^2 + p^\mu p^\nu\right) \delta Z_A$$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\left(A_\nu \partial^2 A^\nu - A_\mu \partial^\mu \partial^\nu A_\nu\right)$$

- Hence: $\underbrace{i\Pi_{\mu\nu}}$ $= \sim\!\!\times\!\!\sim \; +$  at 1-loop

$$= i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2) \qquad\qquad \equiv i\,(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{(1)}(q^2)$$

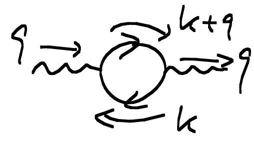
for "1-loop"

and $\delta Z_A = \Pi_{(1)}(0)$.

Thus: We need the coefficient of the log.-divergence of $\Pi_{(1)}(0)$. This gives us

$$c = -\frac{1}{2}\cdot\frac{1}{e^2}\left(\text{coeff. of } \ln\Lambda \text{ in } \Pi_{(1)}(0)\right).$$

11.4 Vacuum polarization in dimensional regularization

Vac. polariz. diagram:  $= i\Pi_{(1)\,\mu\nu}(q^2) = i\,(\eta_{\mu\nu} p^2 - p_\mu q_\nu)\,\Pi_{(1)}(q^2)$

$$i\Pi_{(1)}{}^{\mu\nu}(q^2) = (ie)^2 (-1) \int \frac{d^d k}{(2\pi)^d} \, \mathrm{tr}\left[\gamma^\mu \frac{i}{\not{k}-m} \gamma^\nu \frac{i}{\not{k}+\not{q}-m} \right]$$

run from $0,1,\dots,d-1$ in d dimensions!

(We may think of defining our theory in d dim.-s. from the very beginning)

Note: We will, in the end, treat n as a complex variable. Log. divergences of a diagram will show up as poles in d at certain integer, real values.

Example:
$$\int_1^\infty \frac{x^{d-1}dx}{(x^2)^2} = \int_1^\infty x^{d-5}dx = \frac{1}{d-4}\, x^{d-4}\Big|_1^\infty$$

to avoid IR-divergence (usually absent in case of $m \neq 0$ or external momentum $\neq 0$)

$$= -\frac{1}{d-4} \quad \text{(for } d<4\text{)}$$

common: $d = 4-\epsilon$

$$= \frac{1}{\epsilon}$$

This is ok in the whole compl. d-plane, except at $d=4$.

This corresponds precisely to $\ln\Lambda$ above. In general, we will find $\frac{1}{\epsilon} + \{\text{finite expr. for } \epsilon \to 0\}$.

Crucial advantage: Poincaré-inv. / gauge inv. automatically preserved (this highly non-trivial with a cutoff!)

Let's now calculate:

$$\Pi_{\mu\nu} = (q^2 \eta_{\mu\nu} - q_\mu q_\nu)\Pi$$

$$\Rightarrow \Pi_\mu{}^\mu = (d \cdot q^2 - q^2) \cdot \Pi$$

$$\Rightarrow \Pi_{(1)}(q^2) = \frac{1}{(d-1)\,q^2}\, \Pi_{(1)\mu}{}^\mu(q^2)$$

When calculating $\Pi_{(1)\,\mu}{}^{\mu}$, we encounter

$$\mathrm{tr}\left[\gamma^\mu \frac{i}{\not k - m}\gamma_\mu \frac{i}{\not k+\not q - m}\right] = - \frac{\mathrm{tr}[\gamma^\mu(\not k+m)\gamma_\mu(\not k+\not q+m)]}{(k^2-m^2)((k+q)^2-m^2)} = \dots$$

Use Clifford alg. in d dims. (as usual, $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ with $\mu,\nu = 0 \dots d-1$)

to find: $\gamma^\mu\gamma_\mu = d$; $\gamma^\mu \not k \gamma_\mu = 2\not k - \gamma^\mu\gamma_\mu \not k = (2-d)\not k$

$$= - \frac{\mathrm{tr}[((2-d)\not k + dm)(\not k + \not q + m)]}{(k^2-m^2)((k+q)^2-m^2)} = 4\,\frac{(d-2)\,k\cdot(k+q) - d\,m^2}{(k^2-m^2)((k+q)^2-m^2)}$$

here we used $\mathrm{tr}(\mathbb{1}) = 4$, which in general is non-trivial in dim. reg. (here it's Ok).

Thus:

$$i\Pi_{(1)\,\mu}{}^{\mu}(q^2) = 4e^2 \int \frac{d^nk}{(2\pi)^n}\cdot \frac{(d-2)\,k\cdot(k+q) - d\,m^2}{(k^2-m^2)((k+q)^2-m^2)}$$

Note: One might suspect a quadratic divergence $d = 4$.
In dim. reg., this should show up as a pole in $d = 2$.
However, in the above expression the dangerous term has a prefactor $(d-2)$. Hence $\Pi_\mu{}^\mu$ has no pole at $d = 2$ and hence is <u>not</u> quadratically divergent.

[This is not so easy to see with a cutoff!]

One way to further simplify this integral is via a

Feynman-parameter: $$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA + (1-x)B)^2}$$

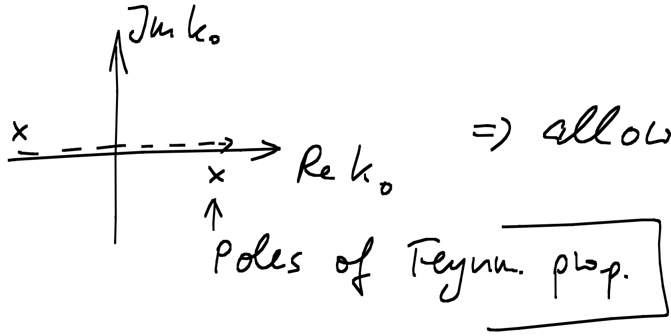
Check this!

$$\Rightarrow i\Pi_{(1)\,\mu}{}^{\mu} = 4e^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \, \frac{(d-2)\,k(k+q) - m^2 d}{\left[(1-x)(k^2-m^2) + x\left((k+q)^2 - m^2\right)\right]^2}$$

Denominator: $$[\cdots]^2 \xrightarrow[k \to k - xq]{} \Big[k^2 + \underbrace{x(1-x)q^2 - m^2}_{\equiv -\Delta}\Big]^2$$

Numerator: $$k(k+q) \xrightarrow[k \to k - xq]{} k^2 + \underbrace{(1-2x)kq}_{\text{vanishes under integration symm. reasons.}} - x(1-x)q^2$$

$$\Rightarrow i\Pi_{(1)\,\mu}{}^{\mu}(q^2) = 4e^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \, \frac{(d-2)(k^2 - x(1-x)q^2) - d \cdot m^2}{(k^2 - \Delta)^2}$$



$\Rightarrow$ allowed contour deformation for k_0-integration:

$$(-\infty \to +\infty) \to (-i\infty \to +i\infty)$$

or: $k_0 \to ik_0$

$$\Rightarrow k^2 = k_0^2 - \vec{k}^2 \longrightarrow -k_E^2 = -k_0^2 - \vec{k}^2$$

euclidean

$$\Rightarrow i\Pi_{(1)\,\mu}{}^{\mu}(q^2) = 4ie^2 \int \frac{d^dk_E}{(2\pi)^d} \int_0^1 dx \, \frac{(d-2)(-k_E^2 + x(1-x)q^2) - m^2 d}{(k_E^2 + \Delta)^2}$$

Calculate, e.g. $\int \frac{d^d k_E}{(2\pi)^d} \cdot \frac{1}{(k_E^2+\Delta)^2} = \underbrace{\int \frac{d\Omega_d}{(2\pi)^d}}_{\substack{\text{well-def.} \\ \text{for all integer} \\ d>1}} \cdot \underbrace{\int_0^\infty dk_E \frac{k_E^{d-1}}{(k_E^2+\Delta)^2}}_{\substack{\text{well-def. for} \\ \text{all } d<4}}$

analytic fct. in d-plane, possibly with poles

$$\cdots = \frac{1}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-d/2)}{\Gamma(2)} \cdot \left(\frac{1}{\Delta}\right)^{2-d/2}$$

has poles at non-positive integers

- expand in ϵ $\quad (d=4-\varepsilon)$
- perform x-integration (easy)

$$\Pi_{(1)}(q^2) = \frac{1}{(d-1)q^2} \Pi_{(1),\mu}{}^{\mu}(q^2) \underset{\substack{\uparrow \\ \text{at } q^2=0 \\ \& \ \varepsilon \to 0}}{=} -\frac{e^2}{6\pi^2\epsilon}$$

Recall: $\quad c = -\frac{1}{2}\cdot\frac{1}{e^2}(\text{coeff. of } \ln\Delta \text{ in } \Pi_{(1)}(0)) = -\frac{1}{2}\cdot\frac{1}{e^2}\frac{(-e^2)}{6\pi^2}$

$$\Rightarrow \quad \beta(e_0) = \frac{e_0^3}{12\pi^2} \quad \text{at 1-loop.}$$

$\Rightarrow$ Landau pole at some (very large) energy scale.

- Can rethink this in terms of renormalized coupling at some non-zero scale μ: $e(\mu)$. It is easy to see that $e(\mu) \approx e_0(\mu)$ (no large logs!). Hence

$$\underline{\beta(e) = \frac{e^3}{12\pi^2} \quad \text{at 1-loop.}}$$