12 Brief introduction to non-abelian gauge theories and the Standard Model

12.1 Non-abelian gauge theories

- Consider a lagrangian with set of fields ψ_i (could also be bosons) transforming in a unitary repres. R of a group G.

- Simplest (and most important) example:

$$G = SU(N) \quad , \quad R = F \text{ (fund. repres.)}$$

$$i = 1 \cdots N \; ; \quad \psi \longrightarrow U\psi \quad (U \in SU(N)) \; ;$$

$$\text{i.e. } \psi'_i = U_{ij}\psi_j .$$

- Demand invariance of $\mathcal{L}$ under this (global) symmetry, e.g.

$$\mathcal{L} = \bar{\psi}_i (i\not{\partial} - m)\psi_i = \underbrace{\bar{\psi}(i\not{\partial}-m)\psi}$$

here ψ denotes a coloumn-vector of Dirac fermions ψ_i and $\bar{\psi}$ denotes a row-vector of fermions $\bar{\psi}_i$ (i.e. a transposition "T" is implicit in $\bar{\psi}$).

$$\Big[\mathcal{L}' = (U\psi)^{+}\gamma^0 (i\not{\partial}-m)U\psi$$
$$= \bar{\psi}U^{+}(i\not{\partial}-m)U\psi$$
$$= \bar{\psi}U^{+}U(i\not{\partial}-m)\psi$$
$$= \mathcal{L}\,, \text{ since } U^{+}U = \mathbb{1} \Big]$$

- To promote this to a local or gauge symm., we need to ensure that $\partial_\mu\psi$ transforms like ψ.

- In analogy to the $U(1)$ case:

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + iA_\mu \quad \text{with } A_\mu \in \text{Lie}(G).$$

(in the $SU(N)$ case, A_μ is a hermitian matrix)

- If T^a is a basis of $Lie(G)$ ($T^a = \frac{1}{2}\sigma^a$, $a = 1,2,3$ for $SU(2)$), we can write $A_\mu = A_\mu^a T^a$. We are dealing with $\dim(Lie(G))$ vector fields!

- Demand $D_\mu' \psi' \overset{!}{=} U(x) D_\mu \psi$, i.e.

$$\underbrace{\partial_\mu (U\psi)}_{(\partial_\mu U)\psi + U\partial_\mu \psi} + iA_\mu' U\psi = U\partial_\mu \psi + U i A_\mu \psi$$

$$\Rightarrow iA_\mu' U = iUA_\mu - \partial_\mu U$$

$$A_\mu' = UA_\mu U^+ + i(\partial_\mu U)U^+$$

or, infinitesimally ($U(x) = e^{iT(x)}$ with infinites. T)

$$\delta A_\mu = i[T, A_\mu] - \partial_\mu T.$$

- Thus: $\bar{\psi}' \not{D}' \psi' = \bar{\psi} U^+ \not{D}' U\psi = \bar{\psi} U^+ U \not{D}\psi = \bar{\psi}\not{D}\psi$

$$\Rightarrow \mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \text{ is invariant.}$$

(more explicitly: $\mathcal{L} = \bar{\psi}_i \left(i\gamma^\mu (\partial_\mu \delta_{ij} + iA_\mu^a (T^a)_{ij}) - m \right)\psi_j$)

- A gauge-inv. kinetic term for A_μ can again be written in analogy to the $U(1)$ case:

$$F_{\mu\nu} \equiv \frac{1}{i}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

($F_{\mu\nu} = F_{\mu\nu}{}^a T^a$ with $F_{\mu\nu}{}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$,

where $[T^a, T^b] = i f^{abc} T^c$ defines the "structure constants" f^{abc}.)

- Since $D_\mu' U = U D_\mu$, we have $D_\mu' = U D_\mu U^+$

and $F_{\mu\nu}' = U F_{\mu\nu} U^+$.

Hence $\mathcal{L} = -\frac{1}{2g^2} \mathrm{tr}\,[F_{\mu\nu} F^{\mu\nu}] + \bar{\psi}(i\not{D} - m)\psi$

is gauge-inv. (If we use the canonical normalization $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$, the fields A_μ^a have a canonical kinetic term — cf. the prefactor $\frac{1}{4}$ of the $U(1)$-case.)

12. The Standard Model

Gauge group: $SU(3) \times SU(2) \times U(1) = G_{SM}$

$$\mathcal{L}_{gauge} = -\sum_{i=1}^{3} \frac{1}{2g_i^2} \mathrm{tr}_i\, F_{\mu\nu}^{(i)} F^{(i)\mu\nu} \qquad i = 1, 2, 3$$

for $U(1), SU(2), SU(3)$.

Fermions: Chirality is crucial! We will work with only left-handed fields, $\psi = \begin{pmatrix} \psi_w \\ 0 \end{pmatrix} = P_L \psi$.

("w" $\to$ Weyl)

(This is conventional, since we could write the same field as a right-handed Dirac spinor:

$$\psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\psi}_w \end{pmatrix}.)$$

The full set of fermions is:

$$\psi = \{ \{ Q^a, U^a, D^a, L^a, E^a \}, \quad a = 1, 2, 3 \}$$

$Q = \begin{pmatrix} u \\ d \end{pmatrix}$ $\qquad$ $L = \begin{pmatrix} \nu \\ e \end{pmatrix}$ $\qquad$ 3 families

[(u, U), (d, D), (e, E) are the l.h./r.h. partners in the familiar Dirac particles up-quark, down-quark, electron.]

These fields transform in differents repr.s of G_{SM}, e.g.

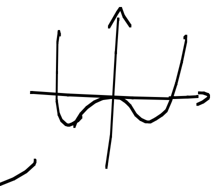
Q is a (fund.!) $SU(3)$-triplet, $SU(2)$-doublet ($\binom{u}{d}$) and has U_1-charge $1/6$, i.e. $D_\mu Q = (\cdots + i\frac{1}{6} A_\mu^{(U(1))} \cdots) Q$.

This completely fixes $\mathcal{L}_{fermions}$, which can very briefly be written as just $\mathcal{L}_{fermions} = \bar{\psi} i \not{D} \psi$

all A's appear; all indices summed.

In addition, there is (complex) scalar $SU(2)$-doublet with $U(1)$-charge $1/2$:

$$\mathcal{L}_{Higgs} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

wrong sign → potential 

$\langle \phi \rangle \neq 0$ in vacuum

$\Downarrow$

Masses for 3 of the 4 gauge fields in $SU(2) \times U(1)$ [the remaining one is the photon] & for fermions, via "Yukawa couplings" of type $\mathcal{L}_{Yukawa} \supset \sim \phi \bar{\psi} \psi$.