3 Noether Theorem

3.1 Motivation

- To fully justify our previously given particle interpretation of the constructed Hilbert space, we would like to define momenta $\hat{p}^i$ (in addition to $\hat{H} = \hat{p}^0$) and evaluate them on n-particle states. (We want to be sure that the index $\bar{p}$ of $a_{\bar{p}}$ etc. is really the momentum.)

- We want to better understand the charge Q given above
- More generally, being able to derive conserved quantities from symmetries is very useful.

3.2 Theorem & Derivation

|| With every continuous symm. of the action comes a conserved current density (and a conserved charge) ||

Symm. trf.: $\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \varepsilon X(x)$,

ε ↑ infinitesimal parameter

such that EOM's don't change. (This is weaker than $S = S'$!)
We assume that this inv. of EOMs is a result of the following trf. property of $\mathcal{L}$:

$\mathcal{L}'(x) = \mathcal{L}(x) + \varepsilon \partial_\mu F^\mu(x)$ [The extra term clearly does not affect EOMs since, to derive the latter, we consider field variations $\delta\varphi$ in a bounded region of $\mathbb{R}^4$. However $\int d^4x\, \partial_\mu \delta F^\mu(x) = 0$ by Gauss' law if δF^μ vanishes at infinity. Hence the extra term does not modify EOMs]

$$\Rightarrow \varepsilon \partial_\mu F^\mu = \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\partial_\mu\varphi$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi$$

Use EOM $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$ & $\delta\varphi = \varepsilon X$

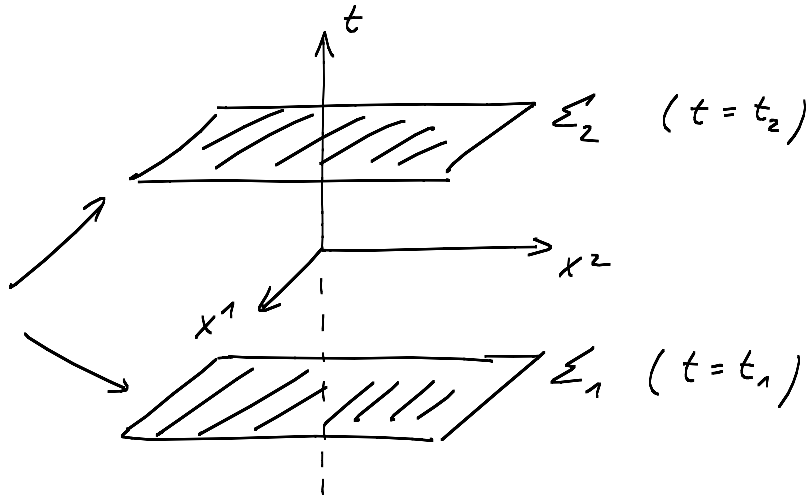
$$\Rightarrow \quad \partial_\mu F^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \cdot X \right)$$

$$\underline{\underline{j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \cdot X - F^\mu}} \quad \text{is conserved:} \quad \partial_\mu j^\mu = 0$$

- Consider $Q(t) \equiv \int d^3x \, j^0(t, \bar{x})$.

$$Q(t_2) - Q(t_1) = \int_{\Sigma_2} df_\mu \, j^\mu - \int_{\Sigma_1} df_\mu \, j^\mu = \int_{Vol.} \partial_\mu j^\mu = 0$$

with



Two (infinite) "spatial slices"

$$\Rightarrow \quad \underline{\underline{\dot{Q} = 0}}$$

Problem: Calculate $\dot{Q}$ directly from its definition (without introducing two different times t_1 & t_2).

Problem: Apply the above reasoning to classical mechanics, where $S = \int dt \, L(q(t), \dot{q}(t))$, derive the Noether theorem and apply it to demonstrate energy conservation.

3.3 Energy-momentum conservation

- Consider translations $\varphi(x) \to \varphi'(x) = \varphi(x+a)$

  a^μ – infinites. 4-vector

  (i.e., we have 4 indep. continuous symms.)

- $\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x) = \mathcal{L}(x) + a^\nu \partial_\mu \underbrace{\left(\eta^\mu{}_\nu \mathcal{L}(x)\right)}_{F^\mu{}_\nu}$

  1st index: 4-vector index, just as for "F^μ" discussed above

  2nd index: labels the 4 different F^μ's associated with the 4 independent a^ν's.

- In general: $j^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \cdot \chi - F^\mu$

- Here: $$j^\mu{}_\nu = \boxed{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \eta^\mu{}_\nu \mathcal{L} \equiv T^\mu{}_\nu}$$

  (energy-mom.-tensor)

  $$\partial_\mu T^{\mu\nu} = 0$$

$\Rightarrow$ 4 conserved currents labelled by ν $\Rightarrow$ 4 conserved charges.

- e.g. $$\int d^3x\, T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\varphi} \dot\varphi - \mathcal{L}\right) = H = P^0 \quad \text{– energy}$$

  $$\int d^3x\, T^{0\nu} = P^\nu \quad \text{– 4-momentum}$$

<u>Problem:</u> Proof that P^ν is a 4-vector, in spite of its apparently non-covariant definition (Idea: Show that P^ν does not change if the space-like hyperplane used in its definition is rotated.)

- After quantization, P^ν are operators (generating translations). Let's calculate P^i (we already know $P^0 = H$) for our real scalar field:

$$P^i = \int d^3x\, T^{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot\varphi} \partial^i \varphi = -\int d^3x\, \pi \vec\nabla \varphi$$

$\downarrow$ problem!

$$P^i = \int \frac{d^3q}{(2\pi)^3}\, q^i \underbrace{a^+_{\vec q} a_{\vec q}}$$

"particle number operator"

"Fourier momentum" of particle

Easy to check! $\hat{P}^\mu |p\rangle = p^\mu |p\rangle$

(Now we have fully justified the particle interpretation of our previously constructed Fock-space.)

Note: for complex scalar: $P^i = \int \frac{d^3q}{(2\pi)^3} q^i (a^+_{\vec q} a_{\vec q} + b^+_{\vec q} b_{\vec q})$

Comment: For a real scalar, we explicitly have

$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}$, which happens to be symmetric in μ, ν. In general (e.g. in QED) our definition does not give a symmetric $T^{\mu\nu}$. However, coupling to gravity requires a symm. $T^{\mu\nu}$. In fact, given some $T^{\mu\nu}$ one can always find a symm. $T'^{\mu\nu}$ with $\int d^3x\, T'^{0\nu} = \int d^3x\, T^{0\nu}$.

An alternative definition, which always gives a symm. $T^{\mu\nu}$, is

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-\det(g_{\delta\sigma})}} \cdot \frac{\delta S}{\delta g^{\mu\nu}(x)}.$$

3.4 Charge of a complex scalar

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$$

- "U(1)-symmetry": $\phi \to \phi' = e^{i\varepsilon}\phi = \phi + i\varepsilon\phi + \cdots$
  $\phi^* \to \phi^{*\prime} = e^{-i\varepsilon}\phi^* = \phi^* - i\varepsilon\phi^* + \cdots$
- $\mathcal{L}' = \mathcal{L} \Rightarrow F^\mu = 0$
- We now want to adapt our general formula

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \chi - F^\mu$$

to the case at hand: 1) $F^\mu = 0$
2) We need to sum over ϕ, ϕ^* (and the corresponding χ, χ^*) for the correct 1st term.

$$\Rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \chi^* \quad \text{with } \chi = i\phi \ ; \ \chi^* = -i\phi^*$$

$$j^\mu = (\partial^\mu \phi^*) i\phi + (\partial^\mu \phi)(-i\phi^*) = \underline{\underline{-i\left(\phi^* \overleftrightarrow{\partial}^\mu \phi\right)}}$$

frequently used notation which implies extra "−" when acting to left

- As usual, $Q = \int d^3x\, j^0$, i.e.

$$Q = -i \int d^3x\ \phi^+ \overleftrightarrow{\partial}_t \phi \quad \text{(after quantization)}$$

$\downarrow$ problem

$$= \int \frac{d^3p}{(2\pi)^3} \left(a^+_{\bar{p}} a_{\bar{p}} - b^+_{\bar{p}} b_{\bar{p}} \right)$$, as claimed previously.