4. Perturbation Theory (Naive Approach)

4.1 Heisenberg picture

Since, in QFT, perturbation theory is usually developed in the "Interaction picture" (which is a "mixture" of Schrödinger- & Heisenberg picture), we first discuss the Heisenberg picture of the free theory:

- So far, our observables are (functions of) $\varphi(\bar{x})$, $\pi(\bar{x})$ $(=\dot{\varphi}(\bar{x}))$, while our states are $|p\rangle = \sqrt{2\omega_{\bar{p}}}\, a^+_{\bar{p}} |0\rangle$

$$|p,q\rangle = \sqrt{2\omega_{\bar{p}}}\sqrt{2\omega_{\bar{q}}}\; a^+_{\bar{p}}\, a^+_{\bar{q}} |0\rangle$$

etc.

- To describe dynamics in the Schrödinger picture, we need to allow for time-evolution of states, e.g.

$$|p_t\rangle \equiv |p_{(0)}\rangle \text{ and } |p_t\rangle = e^{-iHt}|p_{(0)}\rangle \quad \left(\begin{array}{c}\text{in general:}\\ |\psi_t\rangle = e^{-iHt}|\psi\rangle\end{array}\right)$$

We can then analyze the measurement of generic observables $O(\varphi,\pi)$ at some time, e.g. $\langle\psi_t|O|\psi_t\rangle$.

- For free fields, one could easily do that (we have diagonalized H anyway), but even here one can see that this is not natural in a Poinc.-invariant theory:
  - The t-dependence sits in $|\psi\rangle_t$
  - The $\bar{x}$-dependence sits in O (via $\varphi(\bar{x})$ & $\pi(\bar{x})$).

- Better: (Heisenberg picture)

$$\langle\psi_t|O|\psi_t\rangle = \langle\psi|O_t|\psi\rangle \quad \text{with } |\psi\rangle \text{ - time indep.}$$

(in our case $|p\rangle$ etc.)

$$\text{and } O_t = e^{iHt}\,O\,e^{-iHt}$$

- In our case, the crucial time-dep. operator is

$$\varphi(x) = \varphi(t,\bar{x}) = e^{iHt}\int \frac{d^3p}{(2\pi)^3}\cdot\frac{1}{\sqrt{2\omega_{\bar{p}}}}\left(a_{\bar{p}}\, e^{i\bar{p}\bar{x}} + a^+_{\bar{p}}\, e^{-i\bar{p}\bar{x}}\right) e^{-iHt}$$

- To evaluate this, observe that $H a_{\bar{p}} |q_1 \cdots q_n\rangle = a_{\bar{p}}\, (H - E_{\bar{p}})|q_1\cdots q_n\rangle$ since $a_{\bar{p}}$ destroys a particle with energy $E_{\bar{p}} = \omega_{\bar{p}}$.

- Since our multi-particle states are by definition a basis, this holds in complete generality: $\quad H a_{\bar{p}} = a_{\bar{p}}(H - \omega_{\bar{p}})$.

$$\Rightarrow\; e^{iHt} a_{\bar{p}}\, e^{-iHt} = a_{\bar{p}}\, e^{i(H-\omega_{\bar{p}})t} e^{-iHt} = a_{\bar{p}}\, e^{-i\omega_{\bar{p}} t}$$

$$\&\; e^{iHt} a^+_{\bar{p}}\, e^{-iHt} = \quad \cdots \quad = a^+_{\bar{p}}\, e^{i\omega_{\bar{p}} t}$$

$$\Rightarrow\; \varphi(x) = \int \frac{d^3p}{(2\pi)^3}\cdot\frac{1}{\sqrt{2p^0}}\left(a_{\bar{p}}\, e^{-ipx} + a^+_{\bar{p}}\, e^{ipx}\right) \quad \text{with } px = p^0x^0 - \bar{p}\bar{x}$$

$$\& \; p^0 = \omega_{\bar{p}} = E_{\bar{p}}$$

We will now always use this definition of p^0 (unless it is an independent variable).

<u>Note</u>: The same procedure can be used to calculate $\pi(x) = \pi(t,\bar{x})$. We will not really need $\pi(x)$ since it turns out that $\pi(x) = \dot{\varphi}(x)$ holds at the operator level. Furthermore,

$(\Box + m^2)\,\varphi(x)$ holds at the operator level (obvious from the above).

- As an important application, we can establish

<u>Causality</u>:

$[\varphi(x), \varphi(y)] = 0$ if $(x-y)^2 < 0$ [i.e. the measurements of φ at two points with space-like separation do not influence each other]

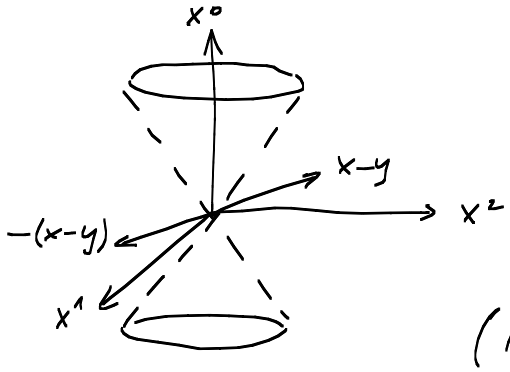
- Demonstration:

$$[\varphi(x), \varphi(y)] = \int \frac{d^3p}{(2\pi)^3\sqrt{2p^0}} \int \frac{d^3q}{(2\pi)^3\sqrt{2q^0}} \left\{ [a_{\bar p}, a^+_{\bar q}]\, e^{-ipx+iqy} + [a^+_{\bar p}, a_{\bar q}]\, e^{ipx-iqy} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3 2p_0}\, e^{ip(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\Big|_{p^0>0} e^{-ip(x-y)} - \int \frac{d^4p}{(2\pi)^3}\, \delta(p^2-m^2)\Big|_{p^0>0} e^{ip(x-y)}$$

- Each term is manifestly invariant under the special Lorentz group (all $SO(1,3)$ trfs. which can be linked to $\mathbb{1}$ by continuous deformations).
- This group allows us to transform $x-y$ (with $(x-y)^2<0$) into $-(x-y)$:



(Problem: prove this!)

- Hence, the two terms cancel and $[\varphi(x), \varphi(y)] = 0$ for $(x-y)^2<0$.

<u>Note:</u> One could also prove "$[\varphi(x), \varphi(y)] = 0$ if $(x-y)^2 < 0$" starting from "$[\varphi(x), \varphi(y)] = 0$ if $x^0 = y^0$", which is known from canonical quantization. The generalization to $x^0 \neq y^0$ can be realized using Lorentz-symm., more specifically:

$$\hat\Lambda\, \varphi(\Lambda^{-1}x)\, \hat\Lambda^{-1} = \varphi(x) \,.$$

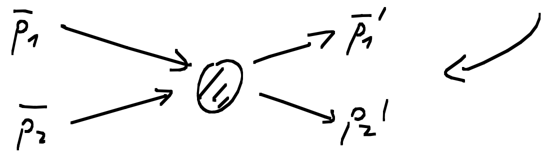
Here $\hat{\Lambda}$ is the repres. of the Lorentz-trf. Λ on the Hilbert space and the equation above is the obvious requirement of Lorentz covariance of our field operators. However, while physically obvious, it <u>does</u> require demonstration (see e.g. Itzykson / Zuber).

4.2 S-Matrix (scattering matrix)

- The simplest modification allowing for scattering is

$$H_0 \longrightarrow H_0 + H_{int} \quad ; \quad H_{int} = \frac{\lambda}{4!}\varphi^4 \supset a^+_{\bar{p}_2'}\, a^+_{\bar{p}_1'}\, a_{\bar{p}_2}\, a_{\bar{p}_1}$$

(equivalently: $\mathcal{L}_0 \to \mathcal{L}_0 + \mathcal{L}$; $\mathcal{L}_{int} = -H_{int}$)



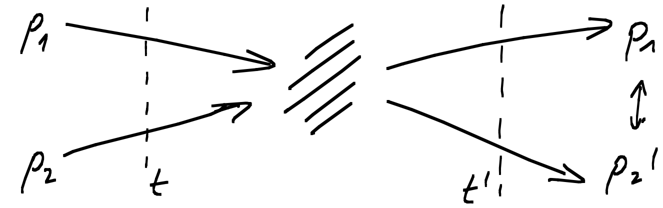
(We exclude a possible term $\sim \varphi^3$
- physically: because $V(\varphi)$ is not bounded from below
- technically: by demanding that $\mathcal{L}$ is symmetric w.r.t. $\varphi \to -\varphi$.)

- For the technical treatment, we work in the "Interaction Picture", in which operators evolve as in the Heisenberg picture, but with the free Hamiltonian, $\quad \mathcal{O}_t = e^{iH_0 t}\,\mathcal{O}_0\, e^{-iH_0 t}$,

  while states carry the non-trivial $H_{int.}$-evolution
$$|\psi_t\rangle = e^{iH_0 t} e^{-iHt} |\psi_0\rangle .$$

- The evolution of an interaction-picture state $|\psi_t\rangle$ into an interaction picture state $|\psi_{t'}\rangle$ is then given by
$$|\psi_{t'}\rangle = e^{iH_0 t'} e^{-iHt'} |\psi_0\rangle = e^{iH_0 t'} e^{-iHt'} \left(e^{iH_0 t} e^{-iHt}\right)^{-1} |\psi_t\rangle$$
$$= \underbrace{e^{iH_0 t'} e^{-iH(t'-t)} e^{-iH_0 t}}\, |\psi_t\rangle$$
$$\equiv \underbrace{\mathcal{U}(t',t)}\, |\psi_t\rangle$$
  This is called the "S-matrix"
  (It simply corresponds to the unitary evolution of states in the interaction picture; cf. time-dependent pert. theory in usual QM)

- Our main physical application will be 2-2-scattering (more generally, 2-n-scattering):



particles separated in space (think of wave-packets)

- The amplitude for this process is given by the projection of $U(t',t)\,|p_1 p_2\rangle$ onto a state $|p_1' p_2'\rangle$ (with different momenta!):

"S-matrix element" $\longrightarrow$ $S_{fi} = \langle p_1' p_2' |\, U(t',t)\, | p_1 p_2 \rangle$

$\uparrow$ final $\uparrow$ initial ,

where $|p_1 p_2\rangle = |p_1 p_2, t\rangle = a^+_{\bar{p}_1} a^+_{\bar{p}_2} |0\rangle$

Here we assume that H_I is irrelevant at early times so that we can use the previously discussed Heisenberg picture

(This is actually not correct (hence "naive approach") since H_I affects even the evolution of a spatially separated single particle. We will treat this properly later on (LSZ-formalism).)

Analogously: $|p_1' p_2'\rangle = a^+_{\bar{p}_1'} a^+_{\bar{p}_2'} |0\rangle$

(Note that we use our "second" normalization convention for the a & a^+.)

- To proceed, we rewrite $U(t',t)$:

$$U(t',t) = e^{iH_0 t'} e^{-iH\cdot(t'-t)} e^{-iH_0 t}$$

$$= e^{iH_0 t'} \underbrace{e^{-iH\Delta} e^{-iH\Delta} \cdots e^{-iH\Delta}}_{n \text{ times}} e^{-iH_0 t} \qquad \left(\Delta = \frac{t'-t}{n}\right)$$

These are products, not arguments!

$$= e^{iH_0 t'} e^{-iH\Delta} \underbrace{e^{-iH_0\cdot(t'-\Delta)} e^{iH_0\cdot(t'-\Delta)}}_{=\mathbb{1}} e^{-iH\Delta} \underbrace{e^{-iH_0\cdot(t'-2\Delta)} \cdots\cdots}_{=\mathbb{1}} \cdots \quad \text{etc.}$$

- Using $H = H_0 + H_{int}$ and the smallness of Δ we write

$$e^{iH_0 t'} e^{-iH\Delta} e^{-iH_0 \cdot (t'-\Delta)} = e^{iH_0 t'} e^{-iH\Delta} e^{iH_0 \Delta} e^{-iH_0 t'}$$

$$\approx e^{iH_0 t'} e^{-iH_{int}\Delta} e^{-iH_0 t'} = e^{-i\underbrace{H_{int.}(t')}\cdot \Delta}$$

time-dependent operator in int.-picture

$$\Rightarrow \quad U(t',t) = e^{-iH_{int}(t')\Delta} e^{-iH_{int}(t'-\Delta)\Delta} \cdots e^{-iH_{int}(t)\Delta}$$

$$= T \exp\left[-i\int_t^{t'} d\tau\, H_{int}(\tau)\right]$$

time-ordering "operator" $\qquad$ H_{int} in interaction picture

(Definition of T: $\quad T\varphi(t_1)\varphi(t_2) = \begin{cases} \varphi(t_1)\varphi(t_2) & \text{if } t_1 > t_2 \\ \varphi(t_2)\varphi(t_1) & \text{if } t_2 > t_1 \end{cases}$)

- Thus, all we need is

$$H_{int}(t) = e^{iH_0 t} \int d^3x \frac{\lambda}{4!} \varphi^4(\bar{x}) e^{-iH_0 t} = \int d^3x \frac{\lambda}{4!} \varphi^4(x) .$$

our familiar space-time dependent quantum field (of the <u>free</u> theory)

- Now we take $t' \to +\infty$ & $t \to -\infty$ and define

$$S = T e^{-i\int dt\, H_{int}(t)} = T e^{i\int d^4x\, \mathcal{L}_{int}(x)}$$

and $\quad S_{fi} = \langle 0 | a_{\bar{p}_1'} a_{\bar{p}_2'} T e^{i\int d^4x \mathcal{L}_{int}} a^+_{\bar{p}_1} a^+_{\bar{p}_2} |0\rangle$

<u>Note</u>: only free fields in this formula!

- Our S-matrix clearly has a trivial part corresponding to the "1" in $e^{i\cdots} = 1 + \cdots$. Hence, we define the transition or T-matrix via

$$S = \mathbb{1} + iT \quad \& \quad S_{fi} = \delta_{fi} + iT_{fi} .$$

- At leading non-trivial order in λ we get

$$iT_{fi} = \langle 0 | a_{\bar{p}_1'} a_{\bar{p}_2'} \left(-\frac{i\lambda}{4!}\right) \int d^4x \, \varphi^4(x) \; a^+_{\bar{p}_1} a^+_{\bar{p}_2} |0\rangle$$

with $\displaystyle \varphi(x) = \int \frac{d^3p}{(2\pi)^3 \, 2p^0} \left(a_{\bar{p}} \, e^{-ipx} + a^+_{\bar{p}} \, e^{ipx} \right)$

and $[a_{\bar{p}}, a^+_{\bar{q}}] = 2p^0 (2\pi)^3 \delta^3(\bar{p}-\bar{q})$. (Note the normalization!)

- T_{fi} can now be worked out by using the commutator to push the a^+'s to the left (the a's to the right) until they annihilate the vacuum. The result (see problems) reads

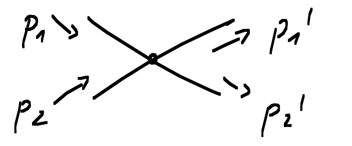
$$iT_{fi} = -i\lambda \, (2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2') .$$

- The momentum-conservation δ-fct. (which always appears in this context) motivates the definition of the "invariant matrix element $\mathcal{M}$:

$$S_{fi} = \mathbb{1} + i(2\pi)^4 \, \delta^4(\cdots) \, \mathcal{M}$$

with $\underline{\underline{i\mathcal{M} = -i\lambda}}$.

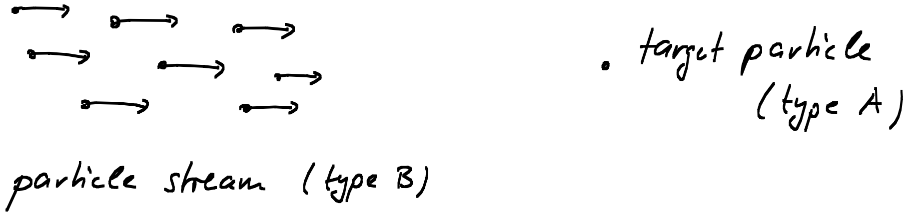
- This is our first "Feynman rule":



"=" $-i\lambda$

4.3 Scattering cross section

Consider fixed-target experiment:



If the beam is homogeneous (transversely) with transverse area F,

$$\frac{N_{events}}{N_B} = \frac{\sigma}{F} \quad \text{or} \quad \sigma = \frac{N_{events}}{(N_B/F)}$$

transverse beam density.

To calculate this, we will need initial states localized in space (at least transversely). Consider wave packets:

$$|f_{\bar{p}}\rangle = \int d\tilde{k}\, f_{\bar{p}}(\bar{k})\,|\bar{k}\rangle \quad \text{with} \quad d\tilde{k} \equiv \frac{d^3k}{(2\pi)^3\, 2k^0}$$

(Here the fct. $f_{\bar{p}}(\bar{k})$ is peaked near $\bar{k} = \bar{p}$; think e.g. of an appropriately shifted Gaussian.)

Normalization: $\langle f_{\bar{p}} | f_{\bar{p}} \rangle = \int d\tilde{k}\, d\tilde{k}'\, f_{\bar{p}}(\bar{k})\, f^*_{\bar{p}}(\bar{k}') \cdot \langle \bar{k}' | \bar{k} \rangle$

$$= \int d\tilde{k}\, |f_{\bar{p}}(\bar{k})|^2 \overset{!}{=} 1$$

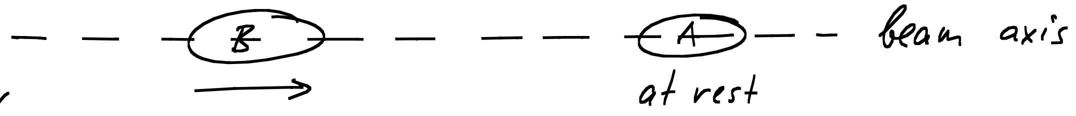
Using $a^+_{\bar{k}} = \int d^3x\, e^{i\bar{k}\bar{x}} \{ k_0 \varphi(\bar{x}) - i\pi(\bar{x}) \}$, we have

$$|f_{\bar{p}}\rangle = \int d^3x \left\{ \int d\tilde{k}\, e^{i\bar{k}\bar{x}} f_{\bar{p}}(\bar{k})\, k^0 \right\} \varphi(\bar{x})|0\rangle$$

$$- i \int d^3x \left\{ \int d\tilde{k}\, e^{i\bar{k}\bar{x}} f_{\bar{p}}(\bar{k}) \right\} \pi(\bar{x})|0\rangle \quad ,$$

which shows that particles can really be localized in $\bar{x}$ while $f_{\bar{p}}$ is appropriately localized in $\bar{k}$. (Recall that the Fourier transform of a Gaussian is again a Gaussian.)

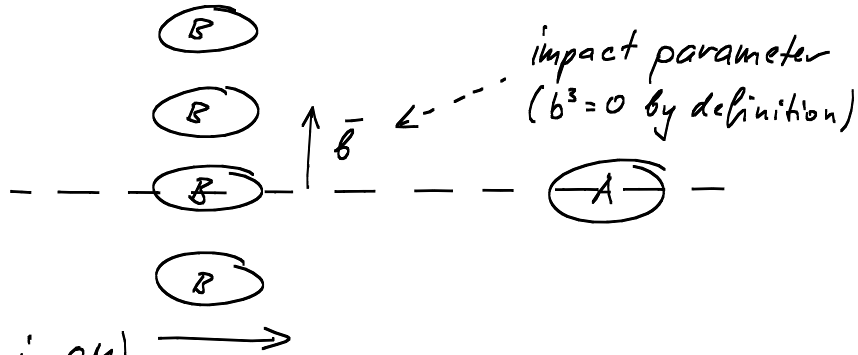
- Thus, we can find functions $f_{\bar{p}_A}$ & $f_{\bar{p}_B}$ characterizing a situation like this:



$$|i\rangle = \int d\tilde{k}_A \, d\tilde{k}_B \; f_{\bar{p}_A}(\bar{k}_A)\, f_{\bar{p}_B}(\bar{k}_B)\, |\bar{k}_A \bar{k}_B\rangle$$

- more realistic:

  - the "shifted" B's can be generated using $\hat{p}^\mu$ (which generates shifts in x'^μ, just as in QM)



- To see this more explicitly, recall: $\quad H a^+_{\bar{k}} = a^+_{\bar{k}}(H + k_0)$

  analogously: $\quad P^i a^+_{\bar{k}} = a^+_{\bar{k}}(P^i + k^i)$.

Thus: $\quad e^{-i\bar{P}\bar{b}}\left(\int d\tilde{k}\, f_{\bar{q}}(\bar{k})\, a^+_{\bar{k}}|0\rangle\right) = \int d\tilde{k}\, f_{\bar{q}}(\bar{k})\, e^{-i\bar{k}\bar{b}}\, a^+_{\bar{k}}|0\rangle$

Using $\quad a^+_{\bar{k}} = \int d^3x\, e^{i\bar{k}\bar{x}}\{\varphi(\bar{x})k^0 - i\pi(\bar{x})\}$

we find

$$\int d^3x \left\{ d\tilde{k}\, f_{\bar{q}}(\bar{k})\, e^{i\bar{k}(\bar{x}-\bar{b})} \right\} \{\varphi(\bar{x})k^0 - i\pi(\bar{x})\}|0\rangle$$

We see that the fct. defining the spatial distribution of the state is indeed shifted by $\bar{b}$.

- Thus, an initial state with fixed impact parameter $\bar{b}$ is

$$|i_b\rangle = \int d\tilde{k}_A\, d\tilde{k}_B\; f_{\bar{p}_A}(\bar{k}_A)\, f_{\bar{p}_B}(\bar{k}_B)\, e^{-i\bar{k}_B\bar{b}}\, |k_A k_B\rangle$$

- Our transition amplitude then reads $\quad \langle p_1 p_2 | S | i_b\rangle$

(We changed $p_1' p_2'$ to $p_1 p_2$ since we now use $p_A p_B$ for the initial momenta) and the corresponding probability naively reads $|\langle p_1 p_2 | S | i_b \rangle|^2$. For many incoming particles, we expect a # of events given by

$$N_{events} = \sum_b |\langle p_1 p_2 | S | i_b \rangle|^2$$

or, in a dense beam, $N_{events} = \frac{N_B}{F} \int d^2 b \; |\langle p_1 p_2 | S | i_b \rangle|^2$

- The corresponding cross-section would be

$$\sigma(p_1, p_2) = \frac{N_{events}}{(N_B / F)} = \int d^2 b \; |\langle p_1 p_2 | S | i_b \rangle|^2$$

- However, this clashes with the δ-function-normalization of $\langle p_1|$, $\langle p_1 p_2|$, etc.. Furthermore, no realistic detector can resolve momenta with infinite precision.

- Hence, we need to sum (integrate) over a small region final-state phase-space:

$$\sigma(V_f) = \int_{V_f} d\tilde{p}_1 \, d\tilde{p}_2 \int d^2 b \; |\langle p_1 p_2 | S | i_b \rangle|^2 \, .$$

- The fact that $\int d\tilde{p}_1 = \int \frac{d^3 p_1}{(2\pi)^3 \, 2 p_1^0}$ is the proper normalization for this integration follows from checking that the probability is 1 for a single particle falling into the right part of phase space:

$$W = \int d\tilde{p} \; |\langle p | f_{\bar{q}} \rangle|^2 = \int d\tilde{p} \; |f_{\bar{q}}(\bar{p})|^2 = 1$$

according to our definition of $f_{\bar{q}}(\bar{k})$. ✓

Instead of working with some $\sigma(V_f)$, we will use a differential formulation

$$d\sigma = \prod_{j=1}^{n} d\tilde{p}_j \int d^2b \; |\langle p_1 \cdots p_n | S | i_b \rangle|^2$$

for n final-state particles.

- With our explicit formula for $|i_b\rangle$ and $S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}$ we find (for non-trivial scattering):

$$d\sigma = \prod_j d\tilde{p}_j \int d^2b \int d\tilde{k}_A \, d\tilde{k}_B \, f_{p_A}(\bar{k}_A) f_{p_A}(\bar{k}_B) \int d\tilde{k}_A' \, d\tilde{k}_B' \, f^*_{p_A}(\bar{k}_A') f^*_{p_B}(\bar{k}_B')$$
$$e^{i\bar{b}(\bar{k}_B' - k_B)} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(p_f - k_i)(2\pi)^4 \delta^4(p_f - k_i') \, ,$$

where
$$p_f = \sum_j p_j$$
$$k_i^{(\prime)} = k_A^{(\prime)} + k_B^{(\prime)} \, .$$

<u>Calculate ...</u>

- $$\int d^2b \, e^{i\bar{b}(\bar{k}_B' - \bar{k}_B)} = (2\pi)^2 \delta^2(k_{B\perp}' - k_{B\perp})$$

- $$\int d^3k_A' \, d^3k_B' \, \delta^4(p_f - k_i') \delta^2(k_{B\perp}' - k_{B\perp}) = \int dk_A^{3\prime} dk_B^{3\prime} \delta(p_f^3 - k_i^{3\prime}) \delta(p_f^0 - k_i^{0\prime})$$

[From now on $k_{B\perp}' = k_{B\perp}$ & $k_{A\perp}' = k_{A\perp}$, where we also used $\delta^2(p_{f\perp} - k_{i\perp})$ from the second δ^4-fct. above.]

$$= \int dk_A^{3\prime} \, \delta(p_f^0 - k_A^{0\prime} - k_B^{0\prime}) \qquad [\text{from now on } k_B^{3\prime} = p_f^3 - k_A^{3\prime}]$$

$$= \int dk_A^{3\prime} \, \delta\Big(p_f^0 - \sqrt{m_A^2 + \underbrace{\bar{k}_A^{\prime 2}}_{k_{A\perp}^2 + (k_A^{3\prime})^2}} - \sqrt{m_B^2 + \underbrace{\bar{k}_B^{\prime 2}}_{k_{B\perp}^2 + (p_f^3 - k_A^{3\prime})^2}}\Big)$$

$$= \frac{1}{\left|\frac{k_A^3}{k_A^0} - \frac{k_B^3}{k_B^0}\right|} \approx \frac{1}{|v_A - v_B|}$$

$k_{A,B} \approx p_{A,B}$ for these purposes

- After all these δ-fct. integrations, $\bar{k}_A'$ & $\bar{k}_B'$ are fixed by

$$k_{A\perp}' = k_{A\perp} \; ; \; k_{B\perp}' = k_{B\perp} \; ; \; p_f^3 - k_A^{3\prime} - k_B^{3\prime} = 0 \; ; \; p_f^0 - k_A^{0\prime} - k_B^{0\prime} = 0 \, .$$

- Together with $p_f^3 - k_A^3 - k_B^3 = 0$ and $p_f^0 - k_A^0 - k_B^0 = 0$ from $\delta^4(p_f - k_i)$, which have not yet used, this is equivalent to

$$\bar{k}_A' = \bar{k}_A \; ; \; \bar{k}_B' = \bar{k}_B \, .$$

- Hence we have

$$d\sigma = \prod_j d\tilde{p}_{ij} \int d\tilde{k}_A d\tilde{k}_B \, |f_{\bar{p}_A}(\bar{k}_A)|^2 \, |f_{\bar{p}_B}(\bar{k}_B)|^2 \, |\mathcal{M}_{fi}|^2 \cdot \frac{1}{4k_A^0 k_B^0 |v_A - v_B|} \cdot (2\pi)^4 \delta^4(p_f - k_i) \, .$$

- We now perform an integration over a sufficiently large path of final phase space to make effectively the $\delta^4(p_f - \cdots)$-fct. very smeared. (But we keep the $d\tilde{p}_{ij}$-notation for simplicity.)

- Now the localization of $f_{\bar{p}_A}$ & $f_{\bar{p}_B}$ is more sharp than the δ^4-fct.. This allows us to perform the $\bar{k}_A$ & $\bar{k}_B$-integrations and simply replace k_i in the δ^4-fct. by p_i.

- Using $\int d\tilde{k}_{A,B} \, |f_{\bar{p}_{A,B}}(\bar{k}_{A,B})|^2 = 1$, we have

$$d\sigma = \underbrace{\frac{1}{4p_A^0 p_B^0 |v_A - v_B|}}_{\substack{\text{prefactor (invariant} \\ \text{under boosts along beam axis)}}} |\mathcal{M}_{fi}|^2 \underbrace{(2\pi)^4 \delta^4(p_f - p_i) \prod_{j=1}^{n} \frac{d^3 p_j}{(2\pi)^3 2p_j^0}}_{n\text{-particle phase space}}$$

which, for $s = (p_A + p_B)^2 \gg m_A^2, m_B^2$,
can be replaced by $\frac{1}{2s}$ ($\sqrt{s} \,\hat{=}$ centre-of-mass energy)

$\Rightarrow$ <u>Highly relativistic case:</u>

$$d\sigma = \frac{1}{2s} |\mathcal{M}_{fi}|^2 \, dX^{(n)}$$

with $(2\pi)^4 \delta^4(\dots)$ & $d\tilde{p}_j$

Problems:

- Demonstrate that $\dfrac{1}{4p_A^0 p_B^0 |v_A - v_B|} = \dfrac{1}{4|p_A^3 p_B^0 - p_B^3 p_A^0|}$ is invariant

under boosts along the beam axis.

Hint: Characterize 4-vectors not by $(p^0, p_\perp, p^3)$, but rather by $(p^+, p^-, p_\perp)$, where $p^\pm \equiv p^0 \pm p^3$.

($\equiv$ light-cone momenta). Work out the transformations of p^+, p^- under boosts along x^3. Show that

$$p_A^+ p_B^- = p_A \cdot p_B + p_A^3 p_B^0 - p_A^0 p_B^3 \qquad (\text{for } p_{A,\perp} = p_{B,\perp} = 0)$$

The claim should then become obvious.

- Show that the above prefactor of the cross section can also be written as $\dfrac{1}{2w(s, m_A^2, m_B^2)}$ [This is most easily done in the target-rest-frame. Cf. Nachtmann's Book.

where $w(x,y,z) \equiv \sqrt{x^2+y^2+z^2-2xy-2xz-2yz}$,

thus bringing the cross section to a manifestly boost-inv. form.

4.4 2-particle phase space and a simple example

- Consider the simple case $A + B \longrightarrow 1 + 2$ in $\lambda\varphi^4$-theory with $s \gg m$. Ignore the matrix element and the prefactor of the cross-section for the moment and focus just on the phase space:

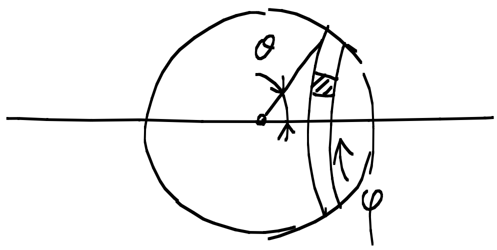
$$\int dx^{(2)} = \int (2\pi)^4 \delta^4(p_1 + p_2 - p_A - p_B) \frac{d^3p_1}{(2\pi)^3 2p_1^0} \cdot \frac{d^3p_2}{(2\pi)^3 2p_2^0}$$

• Perform $\int d^3p_1$ using $\delta^3(\bar{p}_1 + \cdots)$. Working in the cms-system ($\bar{p}_1 = -\bar{p}_2$), we are left with

$$\int \frac{1}{(2\pi)^2 4 p_1^0 p_2^0} d^3\bar{p}_2 \, \delta(\underbrace{p_1^0}_{=|\bar{p}_1| = |\bar{p}_2|} + \underbrace{p_2^0}_{=|\bar{p}_2|} - \sqrt{s}) = \int \frac{d^3\bar{p}_2}{(2\pi)^2 4|\bar{p}_2|^2} \delta(2|\bar{p}_2| - \sqrt{s})$$

Use also $d^3p_2 = d\Omega \, |\bar{p}_2|^2 \, d|\bar{p}_2|$, where

$$d\Omega = d\varphi \sin\theta \, d\theta$$



$$\Rightarrow \int dX^{(2)} = \frac{1}{(2\pi)^2 s} d\Omega \cdot (\sqrt{s}/2)^2 \cdot \frac{1}{2} ,$$

where we decided not to integrate over the angles.

Remembering $|\mathcal{M}|^2 = \lambda^2$, we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{2s} |\mathcal{M}|^2 \cdot \frac{\int dX^{(2)}}{d\Omega} = \frac{\lambda^2}{64\pi^2 s}$$

– indep. of angle! (because of spin-0 particles).

• Note! $\frac{\lambda^2}{s}$ could have been argued just on dimensional grounds.