5 LSZ - Formalism

(Lehmann, Symanzik, Zimmermann ; Nuovo Cimento, 1 ('55) p. 205)

General idea: S-matrix-elements $\longleftrightarrow$ Greens-fcts.

$$ {}_{out}\langle p_1' \cdots p_n' | p_1 \cdots p_m \rangle_{in} \longleftrightarrow \langle 0 | T \varphi(x_1) \cdots \varphi(x_{n+m}) | 0 \rangle $$

(calculable, e.g., from Feynman rules)

5.1 "in" and "out" fields

- We will now work in the Heisenberg picture (not interaction picture!), i.e. $\varphi(x)$ will be the operator version of the field φ of the class. Lagrangian, including full quantum evolution:

$$ \dot{\varphi}(x) = i[H, \varphi(x)] \quad \text{with} \quad H = H_0 + H_{int}. $$

- We consider a theory where $H_{int} \longrightarrow f(t) \cdot H_{int}$, with $f(t) \longrightarrow 0$ for $t \to \pm\infty$ & $f(t) = 1$ at finite t.

("The interactions are switched off adiabatically.")

- We assume that $\varphi(x) \xrightarrow[t \to -\infty]{} Z^{1/2} \varphi_{in}(x)$; $\varphi(x) \xrightarrow[t \to +\infty]{} Z^{1/2} \varphi_{out}(x)$,

where $\varphi_{in/out}$ have the commutation relations of free fields and can be used to define $a_{in/out}$ / $a^{+}_{in/out}$ & states $|p\rangle_{in}$; ${}_{out}\langle p'|$ etc.

[This can only be true in the "weak sense", i.e. for each matrix element of the operator separately. We will demonstrate in a moment that $Z \neq 1$ is necessary for consistency & we will calculate Z later on.]

- The quantum-mechanical amplitude for a scattering process then reads

$$ {}_{out}\langle p_1' \cdots p_n' \,|\, p_1 \cdots p_m \rangle_{in} $$

or $\langle out | in \rangle$ for short.

- We assume the existence of an isomorphism between in & out states:

$$ \varphi_{in}(x) = S\,\varphi_{out}(x)\,S^{-1} \qquad (S \text{ is unitary}) $$

$$ |in\rangle = S\,|out\rangle $$

$$ |out\rangle = S^{-1}|in\rangle , $$

which implies $\langle out | in \rangle = \langle out | S | out \rangle = \langle in | S | in \rangle$.

- Our free fields & multi-particle-states of the last chapter can be identified with either in- or out-states, but not with both. In the last chapter, we have tried to construct S explicitly following the evolution of states in the interaction picture. Here, we will instead directly calculate $\langle out | in \rangle$ in terms of expectation values of interacting fields (Greens fcts.), which we will then calculate in perturbation theory.

5.2 The "Z-factor"

- We perform a simple and more generally useful calculation demonstrating that $Z \neq 1$:

$$ \langle 0 | \varphi(x)\varphi(y) | 0 \rangle = \sum_\alpha \langle 0 | \varphi(x) | \alpha \rangle \langle \alpha | \varphi(y) | 0 \rangle , $$ where α runs over all states (including multi-particle states).

- with $\varphi(x) = e^{i\hat{P}x}\varphi(0)e^{-i\hat{P}x}$ and $e^{-i\hat{P}x}|\alpha\rangle = e^{-ip_\alpha x}|\alpha\rangle$ we find:

(Note: We are in the region where $f(t) = 1$ and Poincare-invariance is OK.)

$$\ldots = \sum_\alpha e^{-ip_\alpha(x-y)} \, |\langle 0|\varphi(0)|\alpha\rangle|^2$$

$$= \int d^4q \sum_\alpha e^{-iq(x-y)} \, |\langle 0|\varphi|\alpha\rangle|^2 \, \delta^4(q-p_\alpha)$$

$$= \int \frac{d^4q}{(2\pi)^3} \, e^{-iq(x-y)} \, \rho(q)$$

where $\rho(q) \equiv (2\pi)^3 \sum_\alpha \delta^4(q-p_\alpha) \, |\langle 0|\varphi|\alpha\rangle|^2$.

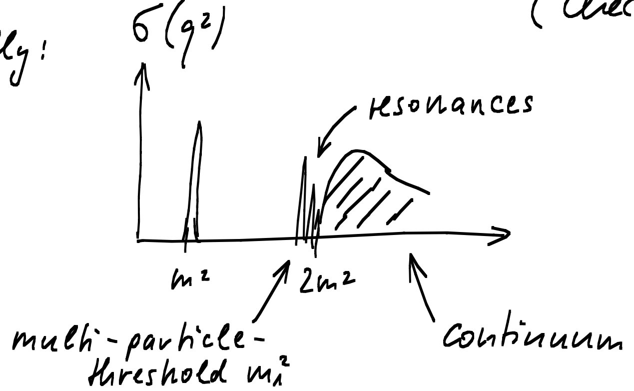
- $\rho(q)$ is manifestly Lorentz-inv. and vanishes for $q^0 < 0$. Thus, we can write it as $\rho(q) \equiv \Theta(q^0)\,\sigma(q^2)$,

  which defines the <u>spectral density</u> $\sigma(q^2)$. It measures the overlap of $\langle 0|\varphi$ with states of mass-squared q^2. For a free theory with mass m, we find

$$\sigma_{free}(q^2) = \delta(q^2 - m^2).$$

(Check this!)

- More generally:



- We can write our result in the form:

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \int_0^\infty dm'^2 \underbrace{\int \frac{d^4q}{(2\pi)^3} \, e^{-iq(x-y)} \, \delta(q^2-m'^2)\,\Theta(q^0)}_{= D(x-y;\, m'^2) \,\equiv\, \langle 0|\varphi_{in}(x)\varphi_{in}(y)|0\rangle} \sigma(m'^2)$$

Check this! free fields with mass m'

- Let us consider the contribution of one-particle-states to our result:

$$\sum_{\substack{\text{one-particle}\\ \text{states } \alpha}} e^{-ip_\alpha(x-y)} \, |\langle 0|\varphi(0)|\alpha\rangle|^2$$

- To evaluate this further, use the time evolution operator U of the interacting theory (defined by $i\partial_t U(t,t_0) = H(t)$, ; $U(t_0,t_0) = \mathbb{1}$).

- More specifically, use $U(t=-\infty, t=0) \equiv U$:

$$|\langle 0|\varphi(0)|\alpha\rangle|^2 = |\underbrace{\langle 0|U}_{=\langle 0| \text{ up to phase}} \underbrace{U^{-1}\varphi(0)U}_{\varphi(x) \text{ at } \bar{x}=0 \,\&\, t=-\infty} \underbrace{U^{-1}|\alpha\rangle}_{=|\alpha\rangle \text{ up to phase (in fact, at } t=+\infty\text{, but that's irrelevant)}}|^2 = \underset{\text{at } t=-\infty}{|\langle 0|\varphi(x)|\alpha\rangle|^2}$$

$$= Z\,|\langle 0|\varphi_{in}(x)|\alpha\rangle|^2 = Z \quad \text{(Check the last equality.)}$$

- Note: The crucial point in the above was that vacuum & 1-part. states go to vac. & 1-part.-st. of the free theory as the interaction is switched on/off.

- Thus, the 1-particle states α contribute a term $Z\,\delta(q^2-m^2)$ to $\sigma(q^2)$ of the interacting theory. Here m^2 is the mass of the 1-particle state of the interacting theory, $m^2 = p_\alpha^2$!

- We find:

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = Z\,D(x-y;m^2) + \int_{m_1^2}^{\infty} dm'^2\,\sigma(m'^2)\,D(x-y;m'^2).$$

- Subtract the same equation with $x \leftrightarrow y$:

$$\langle 0|[\varphi(x),\varphi(y)]|0\rangle = Z\,\Delta(x-y,m^2) + \int_{m_1^2}^{\infty} dm'^2\,\sigma(m'^2)\,\Delta(x-y,m'^2)$$

where $\Delta(x-y,m^2) \equiv \langle 0|[\varphi(x),\varphi(y)]|0\rangle$ in a free theory with mass m.

• Apply $\left.\frac{\partial}{\partial y^0}\right|_{y^0 = x^0}$ $\Rightarrow$ The commutator goes over into

$$[\varphi(x^0, \bar{x}), \pi(x^0, \bar{y})] = i\delta^3(\bar{x}-\bar{y})$$

for both the free and the interacting field (assuming that H_{int} does not involve $\dot{\varphi}$).

$$\Rightarrow \quad 1 = Z + \int_{m_1^2}^{\infty} dm'^2\, \sigma(m'^2)$$

$\Rightarrow$ The difference $1-Z$ accounts for the non-trivial <u>overlap of $\varphi(0)|0\rangle$ with multi-particle states.</u>

$$\Rightarrow \quad 0 \le Z \le 1 \qquad (Z < 1 \text{ if theory not free}).$$

5.3 LSZ reduction formula

$$_{out}\langle p_1' p_2' | p_1 p_2 \rangle_{in} = {}_{out}\langle p_1' p_2' | a^+_{\bar{p}_1, in} | p_2 \rangle_{in}$$

Recall $a^+_{\bar{p}, in} = \int d^3x\, e^{i\bar{p}\bar{x}} \{\varphi_{in}(0,\bar{x})p^0 - i\pi_{in}(0,\bar{x})\}$;

apply $e^{iH_0 t} \ldots e^{-iH_0 t}$ & use $H_0 a^+_{\bar{p}, in} = a^+_{\bar{p}, in}(H_0 + p^0)$

$$\Rightarrow a^+_{\bar{p}, in} = \int_t d^3x\, e^{-ipx} \{\varphi_{in}(x)p^0 - i\pi_{in}(x)\}$$

$$= \int_t d^3x\, e^{-ipx} \frac{1}{i} \overleftrightarrow{\partial_0} \varphi_{in}(x)$$

$$\ldots = \int_t d^3x\, e^{-ip_1 x} \frac{1}{i} \overleftrightarrow{\partial_0}\, {}_{out}\langle p_1' p_2' | \varphi_{in}(x) | p_2 \rangle_{in} \qquad (\text{for } \underline{\text{any}}\ t)$$

$$= \lim_{t \to -\infty} Z^{-1/2} \int_t d^3x\, e^{-ip_1 x} \frac{1}{i} \overleftrightarrow{\partial_0}\, {}_{out}\langle p_1' p_2' | \varphi(x) | p_2 \rangle_{in}$$

Use $\int_{-\infty}^{+\infty} dt \, \frac{\partial}{\partial t} f(t) = \lim_{t \to +\infty} f(t) - \lim_{t \to -\infty} f(t)$.

$$\Rightarrow \quad \cdots = \lim_{t \to +\infty} Z^{-1/2} \int_t d^3x \, e^{-ip_1x} \frac{1}{i} \overleftrightarrow{\partial_0} \, {}_{out}\langle p_1' p_2' | \varphi(x) | p_2 \rangle_{in}$$

$$+ \, i Z^{-1/2} \int d^4x \, \partial_0 \left\{ e^{-ip_1x} \overleftrightarrow{\partial_0} \, {}_{out}\langle p_1' p_2' | \varphi(x) | p_2 \rangle_{in} \right.$$

- 1st term: $\sim \langle p_1' p_2' | a^+_{\bar{p}_1, out} | p_2 \rangle = 0$ if $\bar{p}_1', \bar{p}_2' \neq \bar{p}_1$.
- 2nd term contains

$$\int d^4x \, \partial_0 \, e^{-ip_1x} \overleftrightarrow{\partial_0} \varphi(x) = \int d^4x \left[\left(-\partial_0^2 e^{-ip_1x} \right) \varphi(x) + e^{-ip_1x} \partial_0^2 \varphi(x) \right]$$

can be replaced by $(-\nabla^2 + m^2)$, which can then be moved to $\varphi(x)$ through integration by parts

$$= \int d^4x \, e^{-ip_1x} \left(\Box + m^2 \right) \varphi(x)$$

$$\Rightarrow \quad {}_{out}\langle p_1' p_2' | p_1 p_2 \rangle_{in} = i Z^{-1/2} \int d^4x \, e^{-ip_1x} (\Box + m^2) \underbrace{{}_{out}\langle p_1' p_2' | \varphi(x) | p_2 \rangle_{in}}$$

We now focus on this expression and repeat the above procedure for the particle with momentum p_1': *

$${}_{out}\langle p_1' p_2' | \varphi(x) | p_2 \rangle_{in} = \lim_{y^0 \to +\infty} i Z^{-1/2} \int d^3y \, e^{ip_1'y} \overleftrightarrow{\partial_{y^0}} \, {}_{out}\langle p_2' | \varphi(y) \varphi(x) | p_2 \rangle_{in}$$

Can insert T here without changing the expression.

) The sign is different since $i^* = -i$, where the "" comes from using a instead of a^+.

$$\ldots = \lim_{y^0 \to -\infty} i Z^{-1/2} \int d^3y \, e^{ip_1'y} \overleftrightarrow{\partial}_{y_0} \, {}_{out}\langle p_2' | T\varphi(y)\varphi(x) | p_2 \rangle_{in}$$

$$+ i Z^{-1/2} \int d^4y \, e^{ip_1'y} \left(\Box_y + m^2 \right) {}_{out}\langle p_2' | T\varphi(y)\varphi(x) | p_2 \rangle_{in}$$

- in the first term, $T\varphi(y)\varphi(x)|p_2\rangle_{in} = \varphi(x)\varphi(y)|p_2\rangle_{in}$

  and $\int d^3y \, e^{ip_1'y} \overleftrightarrow{\partial}_{y_0} \varphi(y) |p_2\rangle_{in} \sim a_{\bar{p}_1', in} |p_2\rangle_{in} = 0$

  if $p_1' \neq p_2$.

- Using just the second term we get altogether:

$${}_{out}\langle p_1' p_2' | p_1 p_2 \rangle_{in} =$$

$$= \left(i Z^{-1/2} \right)^2 \int d^4x \int d^4y \, e^{ip_1'y - ip_1x} \left(\Box_y + m^2 \right)\left(\Box_x + m^2 \right) {}_{out}\langle p_2' | T\varphi(x)\varphi(y) | p_2 \rangle_{in}$$

- This procedure can be repeated for p_2, p_2' and can also easily be generalized to more than two initial- and final-state particles. The result reads:

$${}_{out}\langle p_1' \cdots p_n' | p_1 \cdots p_m \rangle_{in} = \left(i Z^{-1/2} \right)^{n+m} \int d^4y_1 \cdots d^4y_n \, d^4x_1 \cdots d^4x_m \cdot$$

$$\cdot \exp i \left(p_1'y_1 + \cdots + p_n'y_n - p_1x_1 - \cdots - p_mx_m \right) \cdot \left(\Box_{y_1} + m^2 \right) \cdots \left(\Box_{y_n} + m^2 \right) \cdot$$

$$\cdot \left(\Box_{x_1} + m^2 \right) \cdots \left(\Box_{x_m} + m^2 \right) \langle 0 | T \varphi(y_1) \cdots \varphi(y_n) \varphi(x_1) \cdots \varphi(x_m) | 0 \rangle$$

- <u>Implication:</u> (to be discussed in more detail later)

  Since, as we will see, Greens-fcts. in momentum space have poles, i.e.
  $$\langle 0 | T\varphi(x) \cdots | 0 \rangle = \int d^4p \, e^{ipx} \frac{1}{p^2 - m^2} \cdots ,$$
  we see that amplitudes are simply given by the residue corresponding to these poles.

5.4 A simple formula for Greens fcts. (= time-ordered vacuum expectation values of fields)

- For simplicity, we restrict ourselves to two fields (the generalization to more fields is obvious)

$$\langle 0|T\varphi(x')\varphi(x)|0\rangle = \langle 0|\varphi(x')\varphi(x)|0\rangle \quad \text{if} \quad x'^0 \equiv t' > t \equiv x^0$$

- For simplicity, we suppress $\bar{x}'$ and $\bar{x}$ and write

$$\cdots = \langle 0|\varphi(t')\varphi(t)|0\rangle = \langle 0|e^{iHt'}\varphi(0)e^{-iH(t'-t)}\varphi(0)e^{-iHt}|0\rangle$$

$$= \langle 0|e^{iHt'}e^{-iH_0t'}\underbrace{e^{iH_0t'}\varphi(0)e^{-iH_0t'}}_{\substack{=\varphi_I(t'),\\ \text{i.e. interaction picture}\\ \text{field, since } \varphi(0)=\varphi_0(0)}}\underbrace{e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}}_{=U(t',t)}\underbrace{e^{iH_0t}\varphi(0)e^{-iH_0t}}_{=\varphi_I(t)}\cdots$$

$$= \langle 0|U(0,t')\varphi_I(t')U(t',t)\varphi_I(t)U(t,0)|0\rangle$$

- Now recall that, in our derivation of the LSZ-reduction formula, we used the Heisenberg picture and, correspondingly, our vacuum $|0\rangle$ is a unique, time-indep. state. [It corresponds to the free in-vacuum at $t=-\infty$, to the full interacting vacuum at finite t, as well as to the free out-vacuum at $t=+\infty$.]
It is clearly <u>not</u> identical to the state $|0\rangle_I$, which we define to have no excitations of the oscillators built from the free field φ_I. However, we want to use this simple vacuum $|0_I\rangle$. Hence the following trick:

$$\langle 0|U(0,t') = \langle 0|U(0,\infty)U(\infty,t') = \langle 0|U(0,\infty)|0\rangle_I\ {}_I\langle 0|U(\infty,t')$$

$\Rightarrow$

$$\langle 0|T\varphi(x')\varphi(x)|0\rangle = \frac{{}_I\langle 0|U(\infty,t')\varphi(t')U(t',t)\varphi(t)U(t',-\infty)|0\rangle_I}{\left({}_I\langle 0|U(0,\infty)|0\rangle_I\,{}_I\langle 0|U(-\infty,0)|0\rangle\right)^{-1}}$$

- The denominator can be rewritten as (Since we are only dealing with phases, $(\cdot\cdot)^{-1} = (\cdot\cdot)^*$)

$$\cdots = {}_I\langle 0|U^{-1}(0,\infty)|0\rangle\langle 0|U^{-1}(-\infty,0)|0\rangle_I$$

$$= {}_I\langle 0|U(\infty,0)U(0,-\infty)|0\rangle_I = {}_I\langle 0|U(\infty,-\infty)|0\rangle_I$$

- Now we recall the formula

$$U(t',t) = T\exp\left[-i\int_{t'}^{t} d\tau\, H_{int}(\tau)\right].$$

Applying this formula we see that, in the numerator, we may as well replace the three T's by a single T in front of the whole expression. Making use of the fact that, under T, we can work with fields like with commuting variables, we find:

$$\langle 0|T\varphi(x')\varphi(x)|0\rangle = \frac{{}_I\langle 0|T\,\varphi_I(x')\,\varphi_I(x)\exp\left[-i\int d\tau\, H_{int}(\tau)\right]|0\rangle_I}{{}_I\langle 0|T\exp\left[-i\int d\tau\, H_{int}(\tau)\right]|0\rangle_I}$$

free vacuum as defined by the free field φ_I.

- This formula holds with as many φ_I – insertions as one wants.