

6 Wick-Theorem & Feynman rules

6.1 Time ordering, normal ordering and the Feynman propagator

- We need to evaluate expressions like

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) e^{i \int d^4x \mathcal{L}_{\text{int}}(x)} | 0 \rangle ,$$

where we have used $-i \int d\tau \mathcal{L}_{\text{int}}(\tau) = i \int d^4x \mathcal{L}_{\text{int}}(x)$

$$\begin{aligned} & \varphi_I^{-}(x) \rightarrow \varphi(x) \\ & | 0 \rangle_I \rightarrow | 0 \rangle \end{aligned} \quad \left. \begin{array}{l} \text{just for notational} \\ \text{simplicity} \\ (\varphi \text{ is still free}) . \end{array} \right\}$$

- Since \mathcal{L}_{int} is a polynomial in φ and since will expand the exponential, we only need to evaluate expressions of the type

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle \text{ with } \underline{\text{free }} \varphi .$$

- Consider first $T \varphi(x) \varphi(y)$ and write

$$\varphi(x) = \int d\vec{k} (a_{\vec{k}}^- e^{-ikx} + a_{\vec{k}}^+ e^{ikx}) = \varphi^v(x) + \varphi^e(x)$$

"Vernichter" "Erzeuger".

- Without loss of generality, let $x^0 > y^0$ such that

$$\begin{aligned} T \varphi(x) \varphi(y) &= \varphi^v(x) \varphi^v(y) + \varphi^v(x) \varphi^e(y) + \varphi^e(x) \varphi^v(y) + \varphi^e(x) \varphi^e(y) \\ &= \varphi^v(x) \varphi^v(y) + \varphi^e(y) \varphi^v(x) + \varphi^e(x) \varphi^v(y) + \varphi^e(x) \varphi^e(y) + [\varphi^v(x), \varphi^e(y)] \end{aligned}$$

- Apply $\langle 0 | \cdots | 0 \rangle$ to find:

$$\langle T \varphi(x) \varphi(y) \rangle = [\varphi^v(x), \varphi^e(y)] \quad (\text{for } x^0 > y^0)$$

(replacing " $| 0 \rangle$ " by " $>$ " for simplicity)

- $\langle T\varphi(x)\varphi(y) \rangle$ for $x^0 < y^0$ follows by symmetry under $x \leftrightarrow y$.
- This expectation value is an important quantity, also known as the Feynman-propagator D_F or a "contraction":

$$\langle T\varphi(x)\varphi(y) \rangle = \overline{\varphi(x)\varphi(y)} = D_F(x-y).$$

- Our previous calculation can be formalised using the concept of "normal ordering":

$\vdots \dots \text{any product of } a\text{'s \& } a^+\text{'s} \dots \vdots = (\text{product of } a^+\text{'s}).(\text{product of } a\text{'s})$

e.g. : $a_{\vec{p}}^- a_{\vec{k}}^+ = a_{\vec{k}}^+ a_{\vec{p}}^-$

This is generally used to symbolize normal-ordered products

- The operation of normal-ordering can obviously be applied to products of φ 's (by writing $\varphi = \varphi^v + \varphi^e$).

- We have already used normal ordering by ignoring the vacuum energy, which just means

$$H_0 = : \frac{1}{2} \int d^3x (\pi^2 + (\bar{\nabla}\varphi)^2 + m^2\varphi^2) :$$

Note that in general: $\langle :\hat{\mathcal{O}}: \rangle = 0$

- Our above result for $T\varphi(x)\varphi(y)$ now reads

$$T\varphi(x)\varphi(y) = : \varphi(x)\varphi(y) : + \overline{\varphi(x)\varphi(y)}.$$

- For notational simplicity, we will write φ_i for $\varphi(x_i)$ etc.
Our result then reads

$$T\varphi_1\varphi_2 = : \varphi_1\varphi_2 : + \overline{\varphi_1\varphi_2}.$$

6.2 Wick theorem

Theorem: $T\varphi_1 \cdots \varphi_n = : \varphi_1 \cdots \varphi_n : + (\text{all contractions of } : \varphi_1 \cdots \varphi_n :)$

Example:

$$\begin{aligned} T\varphi_1 \varphi_2 \varphi_3 \varphi_4 &= : \varphi_1 \varphi_2 \varphi_3 \varphi_4 : + \left(: \overline{\varphi_1} \varphi_2 \varphi_3 \varphi_4 : + 5 \text{ analogous terms} \right) \\ &\quad + \left(: \overline{\varphi_1} \varphi_2 \overline{\varphi_3} \varphi_4 : + 2 \text{ analogous terms} \right) \\ &\quad \swarrow \quad \uparrow \\ \text{Can be dropped} &\quad \text{explicitly!} \\ \text{in this term} &\quad \overbrace{\varphi_1 \varphi_2 \varphi_3 \varphi_4}^{\text{explicitly!}} + \overbrace{\varphi_1 \varphi_2 \overline{\varphi_3} \varphi_4}^{\text{explicitly!}} \\ &= D_F(x_1, x_3) D_F(x_2, x_4) + D_F(x_1, x_4) D_F(x_2, x_3) \end{aligned}$$

Importance of the Theorem: After applying $\langle \dots \rangle$, only the complete contractions survive.

Proof: (by induction)

$n=1$ — trivial

$n=2$ — see above

Step from n to $n+1$: Let $\varphi = \varphi(x)$ and $x^\circ > x_i^\circ$ w.l.o.g.

$$\begin{aligned} T\varphi \varphi_1 \cdots \varphi_n &= \varphi T\varphi_1 \cdots \varphi_n = \varphi : \varphi_1 \cdots \varphi_n : + \varphi (\text{all contractions of } : \varphi_1 \cdots \varphi_n :) \\ &\quad \uparrow \\ &\quad \text{without loss of generality} \end{aligned}$$

Use the Lemma:

$$\varphi : \varphi_1 \cdots \varphi_m : = : \varphi \varphi_1 \cdots \varphi_m : + : \overline{\varphi} \varphi_1 \cdots \varphi_m : + \cdots + : \overline{\varphi} \varphi_1 \cdots \varphi_m :$$

Now the claim for $n+1$ fields (and hence the theorem) clearly follows. \square

Proof of Lemma: let $\varphi = \varphi^e + \varphi^v$, as before.

$$1) \quad \varphi^e : \varphi_1 \cdots \varphi_m : = : \varphi_1 \cdots \varphi_m \varphi_e :$$

$$2) \quad \varphi^v : \varphi_1 \cdots \varphi_m : = : [\varphi^v, \varphi_1] \varphi_2 \cdots \varphi_m : + : \varphi_1 [\varphi^v, \varphi_2] \varphi_3 \cdots \varphi_m : + \cdots \\ \cdots + : \varphi_1 \cdots \varphi_{m-1} [\varphi^v, \varphi_m] : + \underbrace{:\varphi_1 \cdots \varphi_m : \varphi^v}_{=: \varphi_1 \cdots \varphi_m \varphi^v : \text{ as in 1)}}$$

To see this, write $: \varphi_1 \cdots \varphi_m :$ as a sum products of φ_i^e & φ_i^v and apply the formula

$$AB_1 \cdots B_m = [A, B_1] B_2 \cdots B_m + \cdots + B_1 \cdots B_{m-1} [A, B_m] + B_1 \cdots B_m A$$

to each term.

Now we make use of $[\varphi^v, \varphi_i] = [\varphi^v, \varphi_i^e] = \overline{\varphi^v} \varphi_i$ (recall that $x^0 > x_i^0$)

and find

$$\varphi^v : \varphi_1 \cdots \varphi_m : = : \overline{\varphi^v} \varphi_1 \cdots \varphi_m : + \cdots + : \overline{\varphi^v} \varphi_1 \cdots \varphi_m : + : \varphi_1 \cdots \varphi_m \varphi^v :$$

3) Combining the results of 1) & 2), the lemma follows. \square

6.3 The Feynman propagator in more Detail

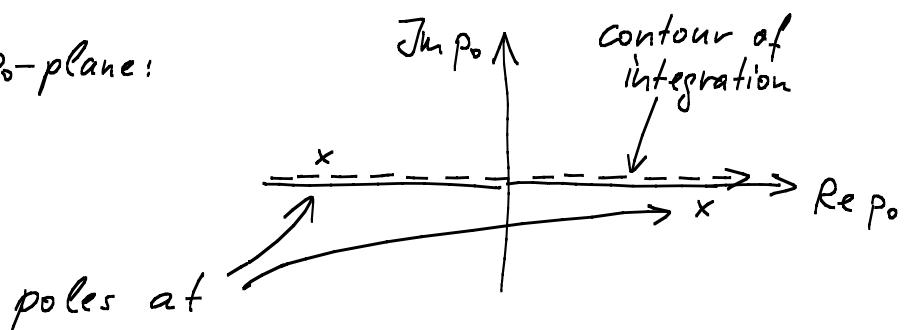
Claim: $D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \cdot \frac{i}{p^2 - m^2 + i\varepsilon} \cdot e^{-ip(x-y)}$

\uparrow
This is just a prescription for dealing with the pole. $\varepsilon \rightarrow 0$ will always be used in the end.

Derivation: let $x^0 > y^0$ (Note that we can use x^0 & x_0 interchangably since $g_{00} = 1$.)

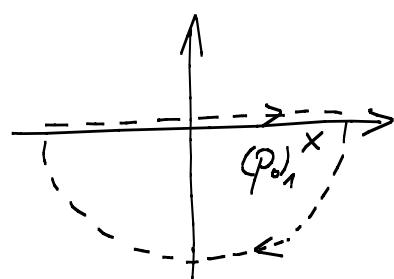
$$D_F(x-y) = \int \frac{d^3 p}{(2\pi)^4} \cdot e^{ip(\bar{x}-\bar{y})} \cdot \int dp_0 \frac{i}{p_0^2 - \bar{p}^2 - m^2 + i\varepsilon} \cdot e^{-ip_0(x_0-y_0)} .$$

go to the complex p_0 -plane:



$$(p_0)_{1,2} = \pm \sqrt{\bar{p}^2 + m^2 - i\varepsilon'} = \pm (\sqrt{\bar{p}^2 + m^2} - i\varepsilon')$$

- Since, for $p_0 \rightarrow -i\infty$, we have $-ip_0(x_0 - y_0) \rightarrow -\infty$, we can close the contour in the lower half-plane picking up just the pole at $(p_0)_1$:



- The residue of

$$\frac{1}{p^2 - m^2} = \frac{1}{(p_0 - (p_0)_1)(p_0 - (p_0)_2)}$$

at $(p_0)_1$ is $\frac{1}{2(p_0)_1}$, giving (now $\varepsilon \rightarrow 0$)

$$D_F(x-y) = \int \frac{d^3 p}{(2\pi)^4} \cdot \frac{i(-2\pi i)}{2(p_0)_1} \cdot e^{-i(p_0)_1(x_0 - y_0) + i\bar{p}(x - \bar{y})}$$

$$= \int d\tilde{p} e^{-i\tilde{p}(x-y)} \quad \text{with } p_0 = \sqrt{\bar{p}^2 + m^2} \text{ as usual.}$$

$$= \langle \varphi(x) \varphi(y) \rangle = \langle T \varphi(x) \varphi(y) \rangle,$$

↑ check this!

- Even though the case $x_0 < y_0$ could be inferred from $x \leftrightarrow y$ symmetry, it is instructive to repeat the analysis. One has to close the contour in the upper half-plane, getting

$$\dots = \int d\tilde{p} e^{-i\tilde{p}(y-x)} = \langle \varphi(y) \varphi(x) \rangle = \langle T \varphi(x) \varphi(y) \rangle. \checkmark$$

6.4 Feynman rules

- We will not give a systematic treatment but work out enough examples to be able to "guess" the general result.

- We start "4-point-functions"

$$G(x_1, \dots, x_4) = \langle T \varphi_1^{\text{int}} \dots \varphi_4^{\text{int}} \rangle$$

interacting fields, to
keep " φ " for free
fields.

$$G(x_1, \dots, x_4) = \langle T \varphi_1 \dots \varphi_4 e^{i \int d^4x \mathcal{L}_{\text{int}}(\varphi)} \rangle / \langle T e^{i \int d^4x \mathcal{L}_{\text{int}}(\varphi)} \rangle$$

- 0. Order in λ :

$$\langle T \varphi_1 \dots \varphi_4 \rangle = \overbrace{\varphi_1 \varphi_2}^1 \overbrace{\varphi_3 \varphi_4}^2 + \text{2 more terms}$$

$$= \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} + \begin{array}{c} 1 \rightarrow 3 \\ | \\ 2 \rightarrow 4 \end{array} + \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array},$$

$$\text{where } \overline{\overbrace{1 \rightarrow 2}} = D_F(x_1 - x_2)$$

- 1. Order in λ :

$$\langle T \varphi_1 \dots \varphi_4 \left(-\frac{i\lambda}{4!} \int d^4x \varphi(x)^4 \right) \rangle = \left(-\frac{i\lambda}{4!} \right) \int d^4x \overbrace{\varphi_1 \varphi_2 \varphi_3 \varphi_4}^1 \overbrace{\varphi_1 \varphi_2 \varphi_3 \varphi_4}^2$$

+ (4!-1) more identical terms

+ terms where $\overbrace{\varphi \varphi}$ appears once

+ terms with $\overbrace{\varphi \varphi} \overbrace{\varphi \varphi}$.

ignoring numerical prefactors
for now

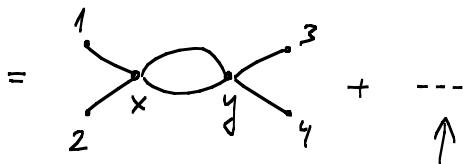
$$= \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} + \begin{array}{c} 1 \rightarrow 3 \\ | \\ 2 \rightarrow 4 \end{array} + \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} + \dots + \begin{array}{c} 1 \\ | \\ 2 \end{array} \begin{array}{c} 3 \\ | \\ 4 \end{array} \infty + \dots$$

- 2. order in λ

$$\langle T \varphi_1 \varphi_2 \varphi_3 \varphi_4 \frac{1}{2!} \left[-\frac{i\lambda}{4!} \int d^4x \varphi(x)^4 \right]^2 \rangle =$$

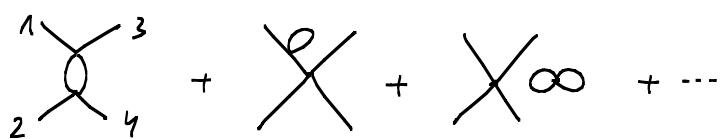
$$= \frac{1}{2!} \left(-\frac{i\lambda}{4!} \right)^2 \sum_{x,y} \overbrace{\varphi_1 \varphi_x \varphi_2 \varphi_x \varphi_x \varphi_y \varphi_x \varphi_y \varphi_y \varphi_3 \varphi_y \varphi_4} + \dots$$

↑
all other full contractions



all other Feynman diagrams built from --- & \times such that each endpoint
 (propagator) (vertex) of each propagator
 is either external
 (points 1-4) or attached to a vertex and
 each external point is the endpoint of
 one of the propagators

- just to give some other full contractions and further diagrams at this order:



... many more!

- It is now easy to guess the Feynman rules (= rules associating analytical expression to each diagram):

---	$= D_F(x-y)$
\times	$= (-i\lambda) \int d^4x$

Combinatorial factors

- The prefactors $\frac{1}{4!}$ are compensated by the $4!$ possibilities to

attach X to

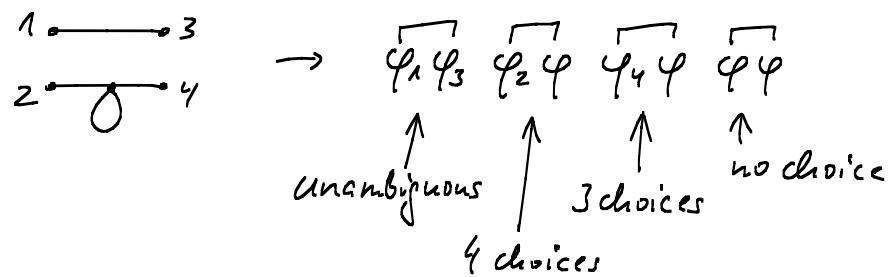
- The prefactor $1/n!$ at n -th order of expanding $\exp(\dots)$ is compensated by the $n!$ possibilities to interchange the X within any given diagram with n vertices.

\Rightarrow generically, there are no combinatorial prefactors

- However, sometimes there are

Symmetry factors

- Example:



$$\Rightarrow \text{prefactor} = \frac{1}{4!} \cdot 4 \cdot 3 = \frac{1}{2}.$$

$$\Rightarrow \left(\begin{array}{c} 1 \xrightarrow{\quad} 3 \\ 2 \xleftarrow{\quad} 4 \end{array} \right) = \frac{1}{2} \left(\begin{array}{l} \text{Result of naive application} \\ \text{of Feynman rules} \end{array} \right)$$

↑

This is called a symm. factor since it is associated with the symmetry of this diagram under the exchange of two of the "open ends" of the vertex.

(At not too high order, symm. factors are easily derived "by hand" on case-by-case basis.)

- So far, we have convinced ourselves that

$$\langle T \varphi_1 \cdots \varphi_n e^{iS_{\text{int}}} \rangle = \left\{ \begin{array}{l} \text{all Feynman diagrams (incl. symm.)} \\ \text{factors} \end{array} \right\}$$

- To calculate greens fcts., we still have to deal with the "denominator" (or normalization factor)

$$\langle T e^{iS_{\text{int}}} \rangle = \left\{ 1 + \text{all Feynman diagrams without external legs} \right.$$

"vacuum bubbles"

$$= 1 + \infty + \infty + \text{Diagram} + \dots$$

should be obvious from our previous discussion

Claim:

$$\langle T \varphi_{1,\text{int.}} \cdots \varphi_{n,\text{int.}} \rangle = C(x_1, \dots, x_n) = \frac{\langle T \varphi_1 \cdots \varphi_n e^{iS_{\text{int.}}} \rangle}{\langle e^{iS_{\text{int.}}} \rangle} =$$

$$= \left\{ \begin{array}{l} \text{all Feynman diagrams (incl. symm. factors)} \\ \text{but without vacuum bubbles} \end{array} \right\}$$

Instead of a proof:

- With any diagram in the numerator which does not contain vacuum bubbles, there come diagrams which differ from it only by the addition of vacuum bubbles, e.g.

$$\begin{array}{ccccccc} 1 & \xrightarrow{\hspace{1cm}} & 3 & + & 1 & \xrightarrow{\hspace{1cm}} & 3 \\ 2 & \xrightarrow{\hspace{1cm}} & 4 & + & 2 & \xrightarrow{\hspace{1cm}} & 4 \\ & & \infty & & & & \infty \\ & & \infty & & & & \infty \end{array} + \dots$$

- One can check that the symm. factors are such that this series can be rewritten as

$\left(\begin{array}{c} 1 \rightarrow 3 \\ 2 \rightarrow 4 \end{array} \right) (1 + \text{all vacuum bubbles (incl. symm factors)})$

↑
including the symm. factor, if there is one

- This works for any diagram, hence our claim.

6.5 Feynman rules in momentum space

"-" for incoming particles
"+" for outgoing particles

- Define $\tilde{G}(p_1, \dots, p_n) = \int d^4x_1 e^{-ip_1 x_1} \dots \int d^4x_n e^{+ip_n x_n} G(x_1 \dots x_n)$

- Writing the propagator as $\int \frac{d^4p}{(2\pi)^4} \cdot \frac{i}{p^2 - m^2 + i\varepsilon} \cdot e^{-ip(x-y)}$,

it is easy to perform all x -integrations and to translate every diagram into an expression w/o x -integrals.

- Simplest example:

$$\int d^4x \int d^4y e^{-ip_1 x + ip_2 y} \int \frac{d^4p}{(2\pi)^4} \cdot \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon} = (2\pi)^4 \delta^4(p_2 - p_1) \cdot \underbrace{\frac{i}{p_1^2 - m^2 + i\varepsilon}}$$

We will call this
the "propagator in momentum
space"

- Next example:

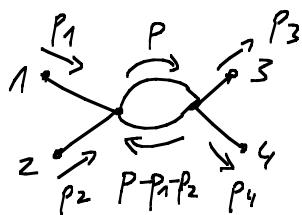
$$\rightarrow -i\lambda \int_{x_1} \dots \int_{x_4} e^{-ip_1 x_1 + \dots + ip_4 x_4} \int_x \left(\int \frac{d^4p}{(2\pi)^4} \cdot \frac{i e^{-ip(x-x_i)}}{p^2 - m^2 + i\varepsilon} \right) (\dots) (\dots) (\dots) (\dots)$$

with x_2, x_3, x_4

$$= -i\lambda \left(\frac{i}{p_1^2 - m^2 + i\varepsilon} \right) \left(\frac{\dots}{p_2} \right) \left(\frac{\dots}{p_3} \right) \left(\frac{\dots}{p_4} \right) \cdot (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2)$$

• Next example:

Fourier-transform of



$$= (-i\lambda)^2 (2\pi)^4 \delta^4(p_4 + \dots - p_1) \left(\frac{i}{p_1^2 - m^2 + i\varepsilon} \right) (\dots) (\dots) (\dots).$$

↑
Symm.-factor

$$\cdot \int \frac{d^4 p}{(2\pi)^4} \cdot \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \frac{i}{(p - p_1 - p_2)^2 - m^2 + i\varepsilon}$$

Problem: Check this (including symm. factor)

From these examples, we "guess" the momentum space Feynman rules:

$$\overrightarrow{p} = \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$\times = -i\lambda$$

$$\& \text{ overall factor } \delta^4(\sum_{\text{outgoing}} p_i - \sum_{\text{outgoing}} p_j)$$

$$\& \int \frac{d^4 p}{(2\pi)^4} \text{ for each closed loop}$$

Note: When drawing the diagram, one has to assign a directed momentum to every line and enforce momentum conservation at every vertex (this implies that one has only one integration per loop)

6.6 Calculating the Z-factor

Before we can apply the above to the calculation of S-matrix

elements, we need to deal with the Z-factor:

- We had derived

$$\langle \varphi^{\text{int}}(x) \varphi^{\text{int}}(y) \rangle = Z D(x-y, m^2) + \int_{m^2}^{\infty} dm'^2 \delta(m'^2) D(x-y, m'^2)$$

where $D(x-y, m^2) = \langle \varphi(x) \varphi(y) \rangle$
 $\uparrow \quad \uparrow$
 free fields with mass m .

- It follows

$$\langle T \varphi^{\text{int}}(x) \varphi^{\text{int}}(y) \rangle = Z D_F(x-y, m^2) + \int_{m^2}^{\infty} dm'^2 \delta(m'^2) D_F(x-y, m'^2)$$

where $D_F(x-y, m^2) = \langle T \varphi(x) \varphi(y) \rangle$
 $\uparrow \quad \uparrow$
 free fields with mass m .

- Fourier transforming & ignoring the Z-fct. we get:

$$\text{---} \circledast \text{---} = \frac{iZ}{p^2 - m^2} + \int_{m^2}^{\infty} dm'^2 \delta(m'^2) \frac{i}{p^2 - m'^2}$$

Symbolic notation for all Feynman diagrams with two external lines (in mom. space; without vac. bubbles;
 without overall Z-fct.)

We will frequently assume that when working with
 Feynman diagrams in mom. space.

- Obvious fact: $\text{---} \circledast \text{---} = \text{---} + \text{---} \text{---} + \text{---} \text{---} + \dots$

Explanation:

\uparrow
 all 1-particle-irreducible diagrams.

1-particle-irred.: --- ; $\text{---} \circlearrowleft$; $\text{---} \text{---}$; ...

1-particle-reducible: $\text{---} \text{---}$; $\text{---} \circledast \text{---}$; ...

Define:

$$\text{Diagram} = -i\Gamma(p^2) \quad \text{"self-energy"}$$

by this we mean $\text{Diagram} / \left(\frac{i}{p^2 - m_0^2} \right)^2$

• Hence:

$$\begin{aligned} \text{Diagram} &= \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2} (-i\Gamma(p^2)) \frac{i}{p^2 - m_0^2} + \dots \\ &= \frac{i}{p^2 - m_0^2 - \Gamma(p^2)} \end{aligned}$$

• It follows that

$$\frac{i}{p^2 - m_0^2 - \Gamma(p^2)} = \frac{iZ}{p^2 - m^2} + \int_{m^2}^{\infty} dm'^2 \delta(m'^2) \frac{i}{p^2 - m'^2}$$

poles must match \Leftarrow no poles at $p^2 < m^2$.
 ↓ ↓

$$m^2 - m_0^2 - \Gamma(m^2) = 0 \quad \underline{\text{This determines } m^2!}$$

also: "residues must match" or, equivalently

$$\frac{i}{p^2 - m_0^2 - \Gamma(p^2)} - \frac{iZ}{p^2 - m^2} \text{ is analytic at } p^2 = m^2.$$

• To check this, write $\Gamma(p^2) = \Gamma(m^2) + \Gamma'(m^2)(p^2 - m^2) + O((p^2 - m^2)^2)$

$$\Rightarrow \frac{i}{p^2 - m^2 - \Gamma'(m^2)(p^2 - m^2) + O((p^2 - m^2)^2)} - \frac{iZ}{p^2 - m^2} \text{ is analytic}$$

$$\Rightarrow \frac{i}{p^2 - m^2} \left(\frac{1}{1 - \Gamma'(m^2) + O(p^2 - m^2)} - Z \right) \text{ is analytic}$$

$$\Rightarrow Z^{-1} = 1 - \Gamma'(m^2) \quad \underline{\text{This determines } Z!}$$

We have learned that

$$\text{---} \text{---} \text{---} \sim \frac{i}{(p^2 - m^2)(1 - \Gamma'(m^2)} \sim \frac{iz}{p^2 - m^2} \quad \text{at } p^2 = m^2$$

This means agreement of pole position
& residue.

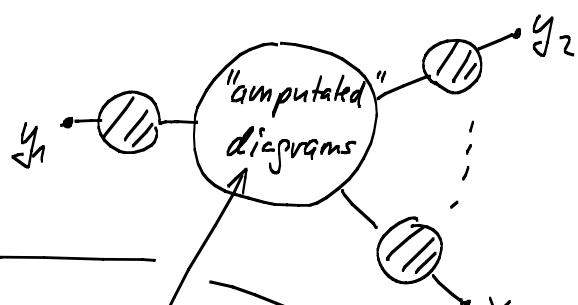
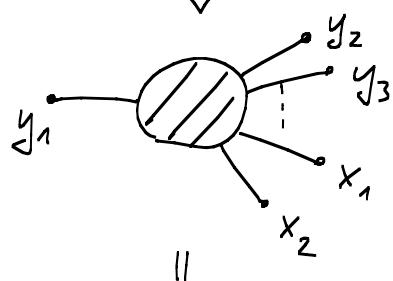
We are now ready to derive the

6.7. Feynman rules for iM_{fi}

Recall that

$$\langle_{\text{out}} p_1' \dots p_n' | p_1 p_2 \rangle_{\text{in}} = (iz^{-1/2})^{n+2} \underbrace{\left[\int_{y_1}^{\rho} e^{ip_1'y_1} (\square_{y_1} + m^2) \right] \left[\dots \right] \dots \left[\dots \right]}_{(n+2) \text{ factors}}.$$

• $\langle T \varphi(y_1) \dots \varphi(y_n) \varphi(x_1) \varphi(x_2) \rangle$



i.e. $- \text{---} \text{---} \text{---} \neq - \text{---} \text{---} \text{---} \text{---} \text{---} \text{---}$ for any A & B

- For each of the variables $y_1 \dots x_2$ we can now perform the calculation

$$(iz^{1/2}) \int_{y_1} e^{ip'_1 y_1} (\square_{y_1} + m^2) \int_{p''_1} \frac{e^{-ip''_1 y_1}}{(2\pi)^4} \cdot \frac{iz}{p''_1 - m^2} (\dots)$$

↓ ↓
 Fourier-trsf. mom.-space freqs.-fct.
 $\square_{y_1} + m^2 \rightarrow - (p''_1 - m^2)$
 $\int_{y_1} \rightarrow (2\pi)^4 \delta^4(p'_1 - p''_1)$
 $\int_{p''_1} \& \delta\text{-fct.} \Rightarrow p''_1 = p'_1$
 $z^{1/2} (\dots)$ with $p''_1 = p'_1$

- Doing this for $y_1 \dots x_2$, we get

$$\langle \underset{\text{out}}{p'_1 \dots p'_n} | \underset{\text{in}}{p_1 p_2} \rangle = \begin{array}{c} p'_1 \\ \vdots \\ p'_n \end{array} \cdot (z^{1/2})^{2+n}$$

amputated, connected* diagrams without vacuum bubbles

* disconnected, e.g. $p_1 - \text{circle} - p'_1$

$p_2 - \text{circle} - p'_2$,

would contradict our assumption of non-trivial scattering.

- Both sides contain $(2\pi)^4 \delta^4(\sum p)$.
- iM_{fi} is defined excluding this factor.
- Hence, recalling our mom.-space Feyn.-rules above, iM_{fi} is calculated according to the following rules:

$$\overleftarrow{\longrightarrow} = \frac{i}{p^2 - m^2 + i\varepsilon} \quad (\text{internal line})$$

$$\overrightarrow{\circlearrowleft} = z^{1/2} \quad (\text{external particle})$$

$$\begin{array}{ccc} p_1 \swarrow \searrow p_3 & = & i\lambda \\ p_2 \nearrow \searrow p_4 & & \end{array} \quad (\text{vertex} : p_4 = p_1 + p_2 - p_3 \text{ must be imposed})$$

$$\int \frac{d^4 p}{(2\pi)^4} \quad \text{for each undetermined momentum} \\ (\text{i.e. each loop})$$

Comment:

- If we now try to calculate cross-sections at an order in λ which allows loops, we find divergent integrals $d^4 p$
- We could analytically continue $p_0 \rightarrow ip_0$ such that $-p^2 > 0$ & introduce a cutoff: $-p^2 < \Lambda^2$.
- We could calculate 2 observables (e.g. m^2 and the $2 \rightarrow 2$ cross sect. at some s) as fcts. of m_0^2 , λ & Λ .
- If we now calculate any further observable, we can express it as a fct. of the first two observables & λ . Now, taking $\Lambda \rightarrow \infty$ gives a finite result.
(Because our λp^4 theory is renormalizable.)
- However λp^4 -theory is not well-suited to explain renormalization. We will first introduce some more realistic & (from the point of view of renormalization) simpler models and return to loop calculations subsequently.