8 Spinors

8.1 Motivation

Lorentz-trf. of fields:

$$\varphi(x) \longrightarrow \varphi(\Lambda^{-1}x)$$
$$A^\mu(x) \longrightarrow \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$
$$F^{\mu\nu}(x) \longrightarrow \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda^{-1}x)$$

etc.

The indices μ, ν can be formally combined into one index i (running from $1, \cdots, 4\cdot 4 = 16$), so that $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \to R(\Lambda)^i{}_j$

Thus, more generally:

$$\phi^i(x) \longrightarrow R(\Lambda)^i{}_j \, \phi^j(x),$$

where $R(\Lambda)$ is a representation of $SO(1,3)$.

[For any group G a repr. is a map $G \to R(G)$, where $R(G)$ is a lin. op. (matrix) on some vector space, satisfying

$$R(\mathbb{1}) = \mathbb{1} \; ; \; R(g\cdot h) = R(g)\cdot R(h) \;]$$

- We already know representations of $SO(1,3)$ on $\mathbb{R}$ or $\mathbb{C}$ (trivial), on $\mathbb{R}^4$ (i.e. on vectors A^μ), on $\mathbb{R}^4 \otimes \mathbb{R}^4$ (i.e. on tensors $t^{\mu\nu}$), on antisymm. tensors $F^{\mu\nu}$ (forming a subspace of $\mathbb{R}^4 \otimes \mathbb{R}^4$), etc.

- As we will now show, there exists a completely different repr. $\Lambda \to S(\Lambda)$ (spinor repres.), which is not just a product of many $\Lambda^\mu{}_\nu$'s. The corresponding fields are called spinors and describe fermions.

8.2 Some mathematical preliminaries

- Groups which are also manifolds (with certain extra conditions) are called <u>Lie groups</u>. Examples are provided by many matrix groups, e.g. $O(n)$ (orthogonal)
  $U(n)$ (unitary)
  etc.

- Such groups have <u>Lie algebras</u> i.e. (specifically for matrices):

G — matrix group
$\mathrm{lie}(G)$ — vector space of matrices

such that
$$\begin{array}{ccc} y & \longmapsto & g = \exp(y) \\ \cap & & \cap \\ \mathrm{lie}(G) & & G \end{array}$$

is differentiable & 1-to-1 in a neighbourhood of $0 \in \mathrm{lie}(G)$ & $\mathbb{1} \in G$.

- <u>Example</u>: $G = SO(3)$; $\mathrm{Lie}(G) = \{$ traceless, antisymm. 3×3 matrices $\}$

indeed: $R \in SO(3)$; $R = \exp(T)$

$$RR^T = \exp(T)\exp(T)^T = \exp(T + T^T) = \mathbb{1}$$

if T is antisymm.

- On $\mathrm{Lie}(G)$ there exists a natural operation

$$a, b \in \mathrm{Lie}(G) \longmapsto \underbrace{[a, b]}_{} \in \mathrm{Lie}(G)$$

This is just the usual commutator of matrices in the case of matrix groups.

- It is natural in the sense that:

$$A \cdot B \cdot A^{-1} \cdot B^{-1} \equiv C \in G \quad (\text{if } A, B \in G)$$

Let $A = e^{\varepsilon a}$; $B = e^{\varepsilon b}$ (ε small)

$$\Rightarrow \left(\mathbb{1} + \varepsilon a + \tfrac{1}{2}\varepsilon^2 a^2\right)(\cdots)(\cdots)(\cdots) + O(\varepsilon^3)$$

$$= \mathbb{1} + \varepsilon^2 [a, b] + O(\varepsilon^3) = C \in G$$

This can only be true if $[a, b] \in \mathrm{Lie}(G)$ (& $C = e^{\varepsilon^2 [a,b]}$).

- For Lie-algebras, one also has the concept of a repres.:

$$\mathrm{Lie}(G) \ni a \longmapsto R(a) \quad (\text{lin. operator or matrix})$$

(a different)

with $0 \longmapsto 0$

$$\& \quad R([a,b]) = [R(a), R(b)].$$

- <u>We need the following crucial fact:</u>

Given some repr. R of $\mathrm{Lie}(G)$, we can always construct a corresponding repr. of G, simply by exponentiation:

$$R(A) = \exp(R(a)) \qquad (\text{if } A = e^a).$$

(More strongly: The Lie-algebra & its repr.s (essentially) determine the Lie-group & its repr.s)

- <u>Idea of proof:</u> • Need to show $R(A) R(B) = R(AB)$
  - For $A = e^a$, $B = e^b$, $AB = C = e^c$, we only need to show: $e^{R(a)} e^{R(b)} = e^{R(c)}$.
  - We know that $e^a e^b = e^c$. ...

- It is also known that

$$e^a e^b = e^{Z(a,b)} \quad \text{with}$$

$$Z(a,b) = a + b + \tfrac{1}{2}[a,b] + \tfrac{1}{12}[a,[a,b]] - \tfrac{1}{12}[b,[a,b]] + \dots$$

involving just commutators!

(Baker - Campbell - Hausdorff).

- Hence

$$e^{R(a)} e^{R(b)} = e^{Z(R(a),R(b))} \underset{\uparrow}{=} e^{R(Z(a,b))} = e^{R(c)} \quad \text{"}\square.\text{"}$$

Here we use the fact that we are dealing with a Lie-alg. representation.

8.3 The spinor representation of SO(1,3)

- $\Lambda \in SO(1,3)$ has to satisfy $\Lambda_\mu{}^\nu \Lambda_\rho{}^\sigma \eta_{\nu\sigma} = \eta_{\mu\rho}$.
- For infinitesimal trfs. we write $\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \underset{\uparrow}{i}\, T_\mu{}^\nu$

"physicist's convention"

$$\Rightarrow (\delta_\mu{}^\nu + i T_\mu{}^\nu)(\delta_\rho{}^\sigma + i T_\rho{}^\sigma)\eta_{\nu\sigma} = \eta_{\mu\rho} + O(T^2)$$

$$T_{\mu\rho} + T_{\rho\mu} = 0$$

(i.e. SO(1,3) is generated by antisymm. matrices (with lower indices))

- Canonical basis: $T_\mu{}^\nu = \underset{\uparrow}{t^{\rho\sigma}} (\underset{\uparrow}{M_{\rho\sigma}})_\mu{}^\nu$

both antisymm. in ρ,σ;

6 parameters; 6 generators

$((16-4)/2 = 6)$

Problem: Define explicitly the canonical basis (such that $M_{\mu\nu}$ generates the rotation in the μ-ν-plane) and show:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}\right)$$

- Any Λ (which can be reached continuously from $\mathbb{1}$) can be written as $\Lambda = \exp(i t^{\mu\nu} M_{\mu\nu})$.

- To define the spinor repr., we first introduce the <u>Clifford algebra</u>, which is generated by $\mathbb{1}$ & 4 elements γ^μ satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

(This algebra is finite-dimensional.)

- Even though a lot more can be done "abstractly", we will immediately give an explicit repr. in terms of 4×4 matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \quad ; \quad \sigma^\mu = (\sigma^0, \sigma^i) = \left(\mathbb{1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\bar\sigma^\mu = (\sigma^0, -\sigma^i).$$

(We will also use $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$.)

- To check that these γ's form a repres. of the Clifford algebra, note:

$$\{\gamma^\mu, \gamma^\nu\} = \left\{\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix}\right\} =$$

$$= \begin{pmatrix} \sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$ and consider explicitly

the cases $\mu, \nu = 0,0 \,/\, 0,i \,/\, i,j$.

- <u>Problem:</u> Prove that

$$\mathcal{M}_{\mu\nu} = + \frac{i}{4} [\gamma_\mu, \gamma_\nu] \text{ satisfy the same commut.}$$

relations as $M_{\mu\nu}$ (i.e. they form a repr. of the Lie-alg. of $SO(1,3)$).

- Thus, with any $\Lambda = e^{it^{\mu\nu}M_{\mu\nu}} \in SO(1,3)$ near $\mathbb{1}$, we can associate an action on $\mathbb{C}^4$: $\psi_D \xrightarrow{\Lambda} \exp\left(it^{\mu\nu}\left(\frac{+i}{4}[\gamma_\mu, \gamma_\nu]\right)\right)\psi_D$ with $\psi_D \in \mathbb{C}^4$.

<u>Thus:</u> A Dirac spinor is a (set of) fields $(\psi_D)_a(x)$ ($a = 1 \cdots 4$) transforming under (small!) Lorentz trfs. as

$$(\psi_D)_a(x) \xrightarrow{\Lambda} S(\Lambda)_{ab}(\psi_D)_b(\Lambda^{-1}x),$$

where $S(\Lambda) = \exp\left(it^{\mu\nu} \frac{+i}{4}[\gamma_\mu, \gamma_\nu]\right)$

(γ_μ is a matrix $(\gamma_\mu)_{ab}$)

for $\Lambda = \exp(it^{\mu\nu}M_{\mu\nu})$.

[<u>Note:</u> The map $\Lambda \to S(\Lambda)$ is <u>not</u> globally defined on $SO(1,3)$. E.g., for a rotation around some fixed axis by φ ($\Lambda = \Lambda(\varphi)$), we can define $S(\Lambda(\varphi))$ near zero and then let φ grow to 2π. We get $\Lambda(2\pi) = \mathbb{1}$ and $S(\Lambda(2\pi)) = -\mathbb{1}$. It would thus be more appropriate to call the trfs. S of a spinor ($S \in Spin(1,3)$) the fund. symm. of space-time and let vectors transform with $\Lambda = \Lambda(S)$. $Spin(1,3)$ is the "double-cover" of $SO(1,3)$.]

- The Dirac spinor is <u>not</u> an irred. repres. since

$$\mathcal{M}_{\mu\nu} = +\frac{i}{4}[\gamma_\mu,\gamma_\nu] = +\frac{i}{4}\left[\begin{pmatrix} 0 & \sigma_\mu \\ \bar\sigma_\mu & 0\end{pmatrix}, \begin{pmatrix} 0 & \sigma_\nu \\ \bar\sigma_\nu & 0\end{pmatrix}\right]$$

$$= +\frac{i}{4}\begin{pmatrix} \sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu & 0 \\ 0 & \bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu\end{pmatrix}$$

$\Rightarrow \quad \psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix}$, where α & $\dot\alpha$ run over 1, 2.

ψ & $\bar\chi$ are Weyl spinors transforming under two different repr.s of $SO(1,3)$ [generated by the above combinations of σ-matrices]. In fact, one is the compl. conj. repr. of the other, but this is not obvious at the moment.

- This decomposition can also be defined more abstractly introducing $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$:
  - It is obvious that $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$. Hence $\gamma^5\mathcal{M}_{\mu\nu} = \mathcal{M}_{\mu\nu}\gamma^5$.
  - I is easy to check that $(\gamma^5)^2 = \mathbb{1}$.
  - Hence $P_L = \frac{1}{2}(\mathbb{1}-\gamma_5)$ & $P_R = \frac{1}{2}(\mathbb{1}+\gamma_5)$ are projection operators on two orthogonal subspaces of $\mathbb{C}^4$.

$$\left(P_L^2 = P_L \;;\; P_R^2 = P_R \;;\; P_L + P_R = \mathbb{1} \right)$$

  - It follows that $\psi_{D,L} = P_L\psi_D$ & $\psi_{D,R} = P_R\psi_D$ transform independently under $SO(1,3)$ (l.h. & r.h. Dirac spinors). Since, in our explicit repres., $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1}\end{pmatrix}$, we have $\psi_{D,L} = \begin{pmatrix}\psi \\ 0\end{pmatrix}$ & $\psi_{D,R} = \begin{pmatrix} 0 \\ \bar\chi\end{pmatrix}$.

- All of this is quite indirect (we always go via the Lie-alg. of $SO(1,3)$ and us the exp-map) and works in complete analogy for $SO(1,d-1)$ if d is even (for d odd it is similar, but without Weyl spinors). However, specifically for $d=4$ a more explicit construction is possible:

Fact: There exists a $2 \to 1$ map from $Spin(1,3) = SL(2,\mathbb{C})$ to $SO(1,3)$:

Let $M \in SL(2,\mathbb{C})$; $v \in \mathbb{R}^4$ and $\hat{v} \equiv v_\mu \sigma^\mu$.
Note that $\hat{v}$ is a generic hermitian 4x4 matrix. Since

$\hat{v}' = M \hat{v} M^+$ is again hermitian, it can be decomposed

as $\hat{v} = v'_\mu \sigma^\mu$. We check:

$$v'^2 = v_0'^2 - \bar{v}'^2 = \det\begin{pmatrix} v_0'+v_3' & v_1'-iv_2' \\ v_1'+iv_2' & v_0'-v_3' \end{pmatrix} = \det \hat{v}'$$

$$= \det M\hat{v}M^+ = \det \hat{v} = \det\begin{pmatrix} v_0+v_3 & \cdots \\ \cdots & \cdots \end{pmatrix} = v^2 .$$

Hence, any M defines, via $\hat{v} \to \hat{v}' = M\hat{v}M^+$, a Lorentz trf. $v_\mu \to v'_\mu = \Lambda_\mu{}^\nu v_\nu$, where Λ is defined by the last equality (to be true for all v & v').

It is clear that $M \mapsto \Lambda$ implies $-M \mapsto \Lambda$, hence "$2\to1$".

- Fact: Our Weyl spinors ψ_α & $\bar{\chi}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}$ transform in the fund. repres. of $SL(2,\mathbb{C})$ and its compl. conj.
(Hence, as we said above, $SL(2,\mathbb{C})$ is the "true" symm. group of space-time.)

Note: Analogous statements hold for $SO(3)$ & $SU(2)$, explaining the existence of ("2-component non-relativistic") spinors in QM.

8.4. Invariants & EOM

- Returning to Dirac-spinors, let us look for Lorentz-singlets to write down Lagrangians:

  (Since we will now mostly use Dirac spinors, we drop the index D and write $\psi_D = \psi$ (do not confuse with Weyl spinors))

- Infinitesimally, $\psi \to (1 + it^{\mu\nu}\mathcal{M}_{\mu\nu})\psi$

$$\psi^+ \to \psi^+(1 - it^{\mu\nu}\mathcal{M}_{\mu\nu}^+).$$

- Since $(\gamma^0)^+ = \gamma^0$ and $(\gamma^i)^+ = -\gamma^i$, we have

$$\mathcal{M}_{0i}^{\ +} = -\mathcal{M}_{0i} \quad , \quad \mathcal{M}_{ij}^{\ +} = \mathcal{M}_{ij}.$$

- Thus, $\psi^+\psi$ is <u>not</u> invariant (this is related to the non-compactness of $SO(1,3)$, which excludes finite-dimens. unitary representations).

- However, we can show that $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+$

$$\& \ \gamma^0\mathcal{M}_{\mu\nu}^+\gamma^0 = \mathcal{M}_{\mu\nu},$$

  and hence

$$\psi^+\gamma^0 \to \psi^+(1 - it^{\mu\nu}\mathcal{M}_{\mu\nu}^+)\gamma^0 = \psi^+\gamma^0(1 - it^{\mu\nu}\gamma^0\mathcal{M}^+\gamma^0)$$

$$= \psi^+\gamma^0(1 - it^{\mu\nu}\mathcal{M}_{\mu\nu}).$$

  Thus, $\psi^+\gamma^0\psi$ is invariant.

- We define $\bar{\psi} \equiv \psi^\dagger \gamma^0$ and write this invariant as

$$\underline{\underline{\bar{\psi}\psi}}\,.$$

- <u>Problem</u>: Check that $[\mathcal{M}_{\mu\nu}, \gamma_\rho] = -(M_{\mu\nu})_\rho{}^\sigma \gamma_\sigma$.
- Using this, we find

$$(1 + i t^{\mu\nu}\mathcal{M}_{\mu\nu})\gamma_\rho(1 - i t^{\mu\nu}\mathcal{M}_{\mu\nu}) = (1 - i t^{\mu\nu} M_{\mu\nu})_\rho{}^\sigma \gamma_\sigma$$

or, by repeated small trfs.,

$$S(\Lambda)\gamma_\rho S(\Lambda^{-1}) = (\Lambda^{-1})_\rho{}^\sigma \gamma_\sigma\,.$$

- Hence, $(\gamma_\rho)_{ab}$ is an <u>invar. tensor</u> of $SO(1,3)$, where ρ is a vector index, a is a spinor index (like that of ψ_a), b is a spinor index (like that of $\bar{\psi}_a$) transforming with $S(\Lambda^{-1})^T$.

- Thus $\bar{\psi}\gamma^\mu\psi$ is a vector and the simplest Lagrangian with a kinetic term is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,.$$

(The i is necessary to make the action real. Check this!)

- To derive EOMs, treat ψ & $\bar{\psi}$ as indep. variables (like ϕ & ϕ^* earlier):

$$0 = \delta_{\bar{\psi}} S = \int d^4x\, \delta_{\bar{\psi}}\, \bar{\psi}(i\underbrace{\not{\partial}}_{\equiv \gamma^\mu\partial_\mu} - m\psi) = \int d^4x\, \delta\bar{\psi}\,(i\not{\partial} - m)\psi$$

$$\Rightarrow (i\not{\partial} - m)\psi = 0 \quad \text{(Dirac eq.)}$$

- Variation w.r.t. ψ gives the hermit. conj. equation $\bar{\psi}(i\overleftarrow{\not{\partial}} - m) = 0$.

Important observation:

$$(-i\not\partial - m)(i\not\partial - m)\psi = (\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu + m^2)\psi$$

$$= \Big(\tfrac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2\eta^{\mu\nu}}\partial_\mu\partial_\nu + m^2\Big)\psi = (\Box + m^2)\psi \underset{\text{since } (i\not\partial - m)\psi = 0}{\overset{\uparrow}{=}} 0$$

Thus: ψ fulfils Dirac eq. $\Rightarrow$ ψ fulfils Klein-Gordon-eq.

$\Rightarrow$ Ansatz: $\psi(x) \sim u(p)\, e^{-ipx}$ $\quad (p^2 = m^2;\ p_0 > 0)$

$$(i\not\partial - m)\psi = 0 \quad \Rightarrow \quad (\not p - m)\, u(p) = 0$$

- Choose frame where $p = (m, \vec{0})$ $\Rightarrow$ $m(\gamma^0 - \mathbb{1})\, u(p) = 0$

$$\Rightarrow \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p) = 0$$

$\Rightarrow$ two lin. indep. solutions, e.g. $u_s \sim \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$ with $s = 1, 2$

& $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- We choose the convenient (see later) normalization

$$u_s(p) = \sqrt{m}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \quad \text{in frame where } p = (m, \vec{0}).$$

Note: $SO(3) \subset SO(1,3)$; $SO(3) \longrightarrow SU(2)$

(see problems)

generator of rot. around axis "j" $\longmapsto \frac{1}{2}\sigma_j$

$$\left[\left(\tfrac{1}{2}\sigma_1\right), \left(\tfrac{1}{2}\sigma_2\right)\right] = i\left(\tfrac{1}{2}\sigma_3\right)$$

$\Rightarrow$ The operators $\frac{1}{2}\sigma_j$ are the (properly normalized) angular momentum operators.

We can check explicitly that the $SO(3) \subset SO(1,3)$ rotations on spinors are

$$\exp i t^{ij} \left(+\tfrac{i}{4}\right) \begin{pmatrix} \sigma_i \bar\sigma_j - \sigma_j \bar\sigma_i & 0 \\ 0 & \bar\sigma_i \sigma_j - \bar\sigma_j \sigma_i \end{pmatrix} = \exp i t^{ij} \varepsilon_{ijk} \left(\tfrac{1}{2} \sigma^k\right),$$

which specifically for rotations by φ around the 3-axis becomes

$$\psi \rightarrow \begin{pmatrix} \exp(i\varphi \tfrac{1}{2}\sigma_3) & 0 \\ 0 & \exp(i\varphi \tfrac{1}{2}\sigma_3) \end{pmatrix} \psi .$$

Here we see explicitly how $SU(2) \subset SL(2,\mathbb{C})$ acts on spinors. This is just as in QM, where rotations around the 3-axis are generated by $\frac{1}{2}\sigma_3$, which obviously has eigenvalues $\pm 1/2$ (hence "spin 1/2").

[More generally, using our notation $\underset{SO(1,3)}{\underset{\cap}{\Lambda}} \longmapsto \underset{SL(2,\mathbb{C})}{\underset{\cap}{M(\Lambda)}}$,

we have
$$S(\Lambda) = \begin{pmatrix} M(\Lambda) & 0 \\ 0 & \varepsilon^{-1} \bar M(\Lambda) \varepsilon \end{pmatrix} \quad \text{with} \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
implying that boosts act differently on the l.h. & r.h. parts, but this is not essential for the moment.]

- In addition to $u(p)e^{-ipx}$, there are also "neg. frequency" solutions, where $e^{-ip^0t} \rightarrow e^{ip^0t}$ (In the scalar case and in the electrodyn. case in Lorentz gauge, this required no extra treatment since the EOM was just the K.-G.-eq. . This is now different!)

- Ansatz: $\psi(x) = v(p) e^{ipx} \quad (p^2 = m^2, p^0 > 0)$

$$(i\not\partial - m)\psi = 0 \quad \Rightarrow \quad (\not p + m) v(p) = 0 \quad , \text{ let } p = (m, \vec 0)$$

$$\Rightarrow \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} v(p) = 0 \quad \Rightarrow \quad v_s(p) = \sqrt{m} \begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}, \quad s = 1,2$$

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

<u>Normalization:</u>

$$\bar{u}_r(p) u_s(p) = 2m\,\delta_{rs} \quad ; \quad \bar{u}_r(p) v_s(p) = 0$$
$$\bar{v}_r(p) v_s(p) = -2m\,\delta_{rs} \quad ; \quad \bar{v}_r(p) u_s(p) = 0$$

[in <u>all</u> Lorentz frames!]

<u>Another useful set of relations:</u>

$$\sum_{s=1}^{2} (u_s(p))_a (\bar{u}_s(p))_b = (\not{p} + m)_{ab}$$

& $\displaystyle\sum_s v_s(p) \bar{v}_s(p) = \not{p} - m$ (in "matrix notation")

- <u>Derivation:</u> $\sum_s u_s(p) \bar{u}_s(p) = \not{p} + m$ is an equality of 4×4 matrices, thus it is sufficient to demonstrate that they act in the same way on the $\mathbb{C}^4$-basis $u_s(p)$, $v_s(p)$ $(s = 1,2)$:

$$\sum_s u_s(p) \bar{u}_s(p) u_r(p) = \sum_s u_s(p)\, 2m\,\delta_{rs} = 2m\, u_r(p)$$

$$\sum_s u_s(p) \bar{u}_s(p) v_r(p) = 0$$

$$(\not{p} + m) u_r(p) = (2m + (\not{p} - m)) u_r(p) = 2m\, u_r(p)$$
$$(\not{p} + m) v_r(p) = 0$$

(by the definition of u & v).

$\square$

- The proof of $\sum_s v_s(p) \bar{v}_s(p) = \not{p} - m$ proceeds analogously.

<u>Note:</u> The signs are easy to memorize:

[$\sum_s u_s(p) \bar{u}_s(p) = \not{p} + m$ is consistent with $(\not{p} - m) u_s(p) = 0$ since $(\not{p} - m)(\not{p} + m) = p^2 - m^2 = 0$.