9 Quantization of Spinors

9.1 Naive attempt with commutators

$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$; The canonical mom. belonging to ψ is

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = \frac{\partial}{\partial \dot{\psi}_a}\left(i\psi^+\gamma^0\gamma^0\dot{\psi}\right) = i\psi^+_a \quad \text{or} \quad \pi = i\psi^+$$

$\uparrow$ interpreted as a row-vector

- We see that π w.r.t. ψ is ψ^+, the "other" field appearing in $\mathcal{L}$. Hence, we don't need another π belonging to ψ^+. This unusual reduction of d.o.f. is due to the linearity of $\mathcal{L}$ in ∂_t (and the fact that linear EOMs require less initial conditions than quadratic EOMs).

cf. Weinberg, Sect. 7

- We have: $\pi = i\psi^+$

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = i\psi^+\dot{\psi} - \psi^+\gamma^0(i\not{\partial} - m)\psi = -\bar{\psi}(i\gamma^i\partial_i - m)\psi.$$
$$= i\pi\gamma^0(i\gamma^i\partial_i - m)\psi$$

- We could attempt to define:

$$[\psi(\bar{x}), \pi(\bar{y})] = [\psi(\bar{x}), i\psi^+(\bar{y})] = i\delta^3(\bar{x}-\bar{y})\cdot \mathbb{1}.$$

- Proceed by "guessing" the expansion of ψ in creation/annih. ops:

$$\psi(x) = \int d\tilde{p}\left(a^s_{\bar{p}}u_s(p)e^{-ipx} + b^{s+}_{\bar{p}}v_s(p)e^{ipx}\right)$$

(by analogy to compl. scalar ; s-summ. implied).

with $[a^r_{\bar{p}}, a^s_{\bar{p}'}] = (2\pi)^3\delta^3(\bar{p}-\bar{p}')\delta^{rs}2p_0$ & same for b.

<u>Consistency check:</u>

$$[\psi(\bar{x}), \psi^+(\bar{y})] = \int d\tilde{p} \int d\tilde{q} \left(e^{i\bar{p}\bar{x} - i\bar{q}\bar{y}} u_s(p) u_r^+(q) [a^s_{\bar{p}}, a^{r+}_{\bar{q}}] \right.$$

(equal time)

$$\left. + e^{-i\bar{p}\bar{x} + i\bar{q}\bar{y}} v_s(p) v_r^+(q) [b^{s+}_{\bar{p}}, b^r_{\bar{q}}] \right).$$

$$= \int d\tilde{p} \left[e^{i\bar{p}(\bar{x}-\bar{y})} (\not{p} - m) - e^{-i\bar{p}(\bar{x}-\bar{y})} (\not{p} + m) \right] \gamma^0$$

$\downarrow$ $\bar{p} \to -\bar{p}$ in second term

$$= \int d\tilde{p} \; e^{i\bar{p}(\bar{x}-\bar{y})} \left((p^0\gamma_0 + p^i\gamma_i - m) - (p^0\gamma^0 - p^i\gamma_i + m) \right) \gamma^0 .$$

This has no chance to work. Lets however change this sign, assuming "$[b, b^+] = -1$", i.e. interchanging the role of b & b^+.

We find: $[\psi(\bar{x}), \psi^+(\bar{y})] = \int d\tilde{p} \, e^{i\bar{p}(\bar{x}-\bar{y})} \cdot 2p^0 \cdot \mathbb{1} = \delta^3(\bar{x}-\bar{y}) \cdot \mathbb{1}$,
as it should be.

- However, another very serious problem appears: Working out H in terms of creation / annih. operators (cf. a similar calculation below), we find:

$$H = \int d\tilde{p} \; p_0 \sum_s \left(a^{s+}_{\bar{p}} a^s_{\bar{p}} - b^{s+}_{\bar{p}} b^s_{\bar{p}} \right)$$

This wrong sign leads to a spectrum not bounded from below, which is inacceptable. (This is <u>not</u> cured by interchanging the roles of b & b^+.)

- It turns out that this problem can only be solved by a rather fundamental change in the whole formalism of canonical quantization: <u>Use anticommutators instead of commutators!</u>

[That this change is really unavoidable for all fields with half-integer spin can be demonstrated rigorously and is the content of the "Spin-Statistics-Theorem" – see e.g. the book by Streater & Wightman: "PCT, Spin and Statistics, and All That".]

9.2 Quantization with anticommutators

- When canonically quantizing fields with half-integer spin, use anticommutators instead of commutators:

$$[\cdots,\cdots] \longrightarrow \{\cdots,\cdots\} \qquad (\text{where } \{A,B\} \equiv AB + BA).$$

- Following the lines of Sect. 9.1, we find:

$$\{\psi(\bar{x}), \pi(\bar{y})\} = \{\psi(x), i\psi^+(\bar{y})\} = i\,\delta^3(\bar{x}-\bar{y})$$

$$\Downarrow$$

$$\{a^r_{\bar{p}}, a^{s+}_{\bar{q}}\} = \{b^r_{\bar{p}}, b^{s+}_{\bar{q}}\} = (2\pi)^3\, 2p_0\, \delta^3(\bar{p}-\bar{q})\,\delta^{rs}$$

(all other anticommutators vanish).

- Calculating H in the usual way, we find

$$H = \int d\tilde{p}\; p_0 \sum_s \left(a^{s+}_{\bar{p}} a^s_{\bar{p}} + b^{s+}_{\bar{p}} b^s_{\bar{p}} \right)$$

$\uparrow$

The sign problem we encountered before is solved since

$$-\, b^s_{\bar{p}} b^{s+}_{\bar{p}} = +\, b^{s+}_{\bar{p}} b^s_{\bar{p}} + (\text{term} \sim \mathbb{1}),$$

because b & b^+ <u>anti</u>commute.

- As an example calculation, we discuss the derivation of this form of H in some detail:

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, (\pi\dot{\psi} - \mathcal{L}) = \cdots$$

$$= \int d^3x \left[i\psi^+\dot\psi - i\psi^+\dot\psi + \bar\psi\,(-i\underbrace{\bar\gamma}_{\gamma^i}\,\underbrace{\bar\nabla}_{\frac{\partial}{\partial x^i}} + m)\,\psi \right]$$

$$= \int d^3x\ \bar\psi\,(-i\bar\gamma\bar\nabla + m)\,\psi\,.$$

- Using $\psi(\bar x) = \int d\tilde p\,\left(a^s_{\bar p}\,u_s(p)\,e^{i\bar p\bar x} + b^{s+}_{\bar p}\,v_s(p)\,e^{-i\bar p\bar x}\right)$

$$\bar\psi(\bar x) = \int d\tilde p'\left(a^{s'+}_{\bar p'}\,\bar u_{s'}(p')\,e^{-i\bar p'\bar x} + b^{s'}_{\bar p'}\,\bar v_{s'}(p')\,e^{i\bar p'\bar x}\right)$$

as well as $\int d^3x\, e^{i\bar p\bar x \pm i\bar p'\bar x} = (2\pi)^3\,\delta^3(\bar p \pm \bar p')$,

we arrive at an expression with terms of the type

$$a^+a\ ;\ a^+b^+\ ;\ ba\ ;\ bb^+\,.$$

- Focuss first on the type a^+a:

$$H = \int \frac{d\tilde p}{2p_0}\ a^{s'+}_{\bar p}\,a^s_{\bar p}\ \bar u_{s'}(p)\,(\bar\gamma\bar p + m)\,u_s(p) + \underbrace{\cdots}_{\text{3 other types}\,\ldots}$$

- Use $0 = (\not p - m)\,u(p) = (\gamma^0p^0 - \bar\gamma\bar p - m)\,u(p)$

$$\Rightarrow (\bar\gamma\bar p + m)\,u(p) = \gamma^0p^0\,u(p)$$

$$\Rightarrow H = \int \frac{d\tilde p}{2}\ a^{s'+}_{\bar p}\,a^s_{\bar p}\ \bar u_{s'}(p)\,\gamma^0\,u_s(p) + \cdots$$

<u>Technical aside:</u> Using $\bar u(p)(\not p - m) = 0$, $(\not p - m)\,u(p) = 0$,

one can show that $\bar u_r(p)\gamma^0 u_s(p) = \frac{1}{2m}\,\bar u_r(p)\,\{m,\gamma^0\}\,u_s(p)$

$$= \frac{1}{2m}\,\bar u_r(p)\,\{\not p - m + m, \gamma^0\}\,u_s(p) = \frac{1}{2m}\,\bar u_r(p)\,\{\not p,\gamma^0\}\,u_s(p)$$

$$= \frac{p_0}{m}\,\bar u_r(p)\,u_s(p) = \frac{p_0}{m}\,2m\,\delta_{rs} = 2p_0\,\delta_{rs}$$

(analogously: $\bar v_r(p)\gamma^0 v_s(p) = 2p_0\,\delta_{rs}$)

$$\Rightarrow \quad H = \int d\tilde{p}\, p_0\, a_{\bar{p}}^{s+} a_{\bar{p}}^{s} \quad + \cdots$$

Let us now turn to the remaining terms:

- the a^+b^+ & ba terms vanish
- the bb^+ term is treated analogously to the a^+a term:

$$H = \cdots + \int \frac{d\tilde{p}}{2p_0}\, b_{\bar{p}}^{s'} b_{\bar{p}}^{s+}\, \bar{v}_{s'}(p)\,(-\bar{\gamma}\bar{p}+m)\, v_s(p) \quad ;$$

Use $0 = (\not{p}+m)v(p)$ and the "technical aside" above to show that

$$\bar{v}_r(p)(-\bar{\gamma}\bar{p}+m)v_s(p) = \bar{v}_r(p)(-p^0\gamma^0)v_s(p)$$

$$= \bar{v}_r(p)\,\frac{p^0}{2m}\left\{(\not{p}+m-m),\gamma^0\right\}v_s(p) = \frac{p^0}{2m}\,\bar{v}_r(p)\left\{\not{p},\gamma^0\right\}v_s(p)$$

$$= \frac{(p^0)^2}{m}\,\bar{v}_r(p)v_s(p) = -2(p^0)^2\,\delta_{rs}$$

This is where the crucial minus comes in, which potentially causes a problem.

$$\Rightarrow \quad H = \int d\tilde{p}\, p^0\left(a_{\bar{p}}^{s+}a_{\bar{p}}^{s} - b_{\bar{p}}^{s}b_{\bar{p}}^{s+}\right)$$

$$= \int d\tilde{p}\, p^0\left(a_{\bar{p}}^{s+}a_{\bar{p}}^{s} + b_{\bar{p}}^{s+}b_{\bar{p}}^{s}\right) + \text{irrelev. constant}$$

The anticommut. relations save us!

This suggests a particle / antiparticle interpretation of the Fock space

$$|0\rangle \;;\; a_{\bar{p}}^{s+}|0\rangle \;;\; b_{\bar{p}}^{s+}|0\rangle \;;\; a_{\bar{p}}^{s+}a_{\bar{q}}^{r+}|0\rangle \;;\; \text{etc.}$$

defined, as before, by $a_{\bar{p}}^{s}|0\rangle = b_{\bar{p}}^{s}|0\rangle = 0$.

Namely: a^+/a creat. / annih. particles
b^+/b — " — antiparticles.
(each comes with spin $\pm 1/2$, labelled by "s, r" etc.)

Due to the anticommut. relations, these particles are <u>fermions</u> (hence "spin-statistics-theorem"):

$$(a_{\bar{p}}^{s+})^2 |0\rangle = 0 \quad ,$$ i.e. in multiparticle states no particles with identical "labels" ($\bar{p}$ & s) can appear.

[To connect this to the (experimentally well-established) exclusion principle for fermions, one has to be more careful and work with localized states (wave packets) since, for plane waves in infinite space, $\bar{p} = \bar{p}'$ "never occurs".]

9.3 Dirac propagator

(We will be very brief since all calculations are very close to the bosonic case, except for signs and indices (the a of ψ_a).)

- As before, expressions of the type

$$\langle 0| T \,(\text{product of } \psi\text{'s} \,\&\, \bar{\psi}\text{'s}) |0\rangle \qquad (\text{i.e. Greens fcts.})$$

can be evaluated by (anti-!) commuting a, b to the right and a^+, b^+ to the left.

- T is now defined as (with or without "$-$")

$$T \overset{(-)}{\psi}_{a_1}(x_1) \cdots \overset{(-)}{\psi}_{a_n}(x_n) = \text{sgn}(p) \cdot \overset{(-)}{\psi}_{a_{p(1)}}(x_{p(1)}) \cdots \overset{(-)}{\psi}_{a_{p(n)}}(x_{p(n)})$$

where p is a permutation of $\{1 \cdots n\}$ ensuring

$x^0_{p(1)} \leq \cdots \leq x^0_{p(n)}$. $\mathrm{sgn}(p)$ is $+/-$ depending on whether p is even / odd.

- This extra sign in the definition of T does not affect the way in which T appears in the LSZ-formalism (since one always starts with a single product with some order of factors), except for a possible overall sign.

- However, it is only with this def. of T that we can prove the Wick theorem for spinor fields:

$$T\,(\text{product of } \psi\text{'s } \& \ \bar\psi\text{'s})$$
$$= \ :\!\left(\text{product of } \psi\text{'s } \& \ \bar\psi\text{'s} \ + \ \text{all possible contractions}\right)\!:$$

Here, each contraction is defined to include a sign which is the product of all signs arising in the necessary exchanges of ψ's to put contracted pairs next to each other.

$$(\text{e.g.}\ :\overbrace{\psi_1 \psi_2 \bar\psi_3}\bar\psi_4 \ = \ - \overbrace{\psi_1 \bar\psi_3} :\psi_2 \bar\psi_4:)$$

- A contraction is defined as

Dirac propagator

$$\left(\overbrace{\psi_1 \bar\psi_2}\right)_{ab} = \overbrace{\psi_a(x_1) \bar\psi_b(x_2)} = \langle T \psi_a(x_1) \bar\psi_b(x_2) \rangle \equiv S_F(x_1 - x_2)_{ab}.$$

- It can be evaluated (as in the bosonic case, just with a bit more algebra) to give

$$S_F(x_1 - x_2)_{ab} = \int \frac{d^4p}{(2\pi)^4} \cdot \frac{i(\not{p} - m)}{p^2 - m^2 + i\varepsilon}\, e^{-ip(x_1 - x_2)}.$$

Note: No $\psi\psi$ or $\bar\psi\bar\psi$ contractions!

Comment:

The scalar (Feynman) propagator contains the "inverse momentum-space Klein-Gordon operator $\frac{1}{p^2-m^2+i\varepsilon}$,

consistent with the fact that

$$-(\Box_x+m^2)\left(\int\frac{d^4p}{(2\pi)^4}\cdot\frac{i}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}\right) = i\delta^4(x-y)$$

[This also explains the term Greens fct. familiar from class. electrodynamics and provides an alternative route to the derivation of the Feynman propagator. The "$i\varepsilon$" makes it the time-ordered or Feynman propagator. Other propagators, corr. to "advanced" or "retarded" Greens fcts. also exist.]

Analogously, the Dirac propagator fulfils

$$(i\not{\partial}-m)_x\int\frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

$$=\int\frac{d^4p}{(2\pi)^4}\cdot\frac{i(\not{p}+m)(\not{p}-m)}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)} = \mathbb{1}\cdot\delta^4(x-y)\,.$$

Correspondingly, it can be viewed as containing the inverse of the mom.-space Dirac operator: $\frac{1}{\not{p}-m}=\frac{\not{p}+m}{p^2-m^2}$.

9.4 U(1)-symm. of the Dirac Lagrangian

$\mathcal{L}=\bar{\psi}(i\not{\partial}-m)\psi$; $\psi\to e^{-i\varepsilon}\psi$ is a (global) symm.

infinitesimally: $\psi(x) \longrightarrow \psi'(x) = \psi(x) - i\varepsilon\psi(x)$.

$$\boxed{\begin{array}{l} \underline{\text{Noether theorem:}} \quad \varphi \to \varphi + \varepsilon X \quad \text{is symm. of action} \\ \qquad \Rightarrow j^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \cdot X - \underset{\substack{\uparrow \\ \text{from change of } \mathcal{L} \text{ by } \varepsilon \partial_\mu F^\mu}}{F^\mu} \end{array}}$$

In our case: $j^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \cdot (-i\psi) = \bar{\psi} i\gamma^\mu(-i\psi) = \bar{\psi}\gamma^\mu\psi$

(This will be the electromag. current after coupling to the electromag. field)

<u>charge:</u> $Q = \int d^3x\, j^0 = \int d^3x\, \psi^+\psi = \int d\tilde{p} \sum_s \left(a_{\vec{p}}^{s+} a_{\vec{p}}^s - b_{\vec{p}}^{s+} b_{\vec{p}}^s\right)$,

i.e. we have defined particles / antiparticles to have charge ± 1 under this symm.