Quantum Field Theory II

1 Path integral or functional integral approach

1.1 Path integral in quantum mechanics (Feynman, based on first ideas by Dirac)

- Consider a single q.m. particle in a potential (in one dim.):

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \quad ; \quad [\hat{q}, \hat{p}] = i \qquad (\hbar = 1)$$

- The most common explicit realization of the Hilbert space (carrying a repres. of the $\hat{p}$-$\hat{q}$-algebra or Heisenberg algebra) is that by square-integrable functions

$$\psi : q \to \psi(q) .$$

- We will also denote this fct., viewed as an element of the Hilbert space, by $|\psi\rangle$.

- The Heisenberg algebra is represented through

$$\hat{q} : \quad \psi(q) \longrightarrow q \cdot \psi(q)$$
$$\hat{p} : \quad \psi(q) \longrightarrow -i\psi'(q) \qquad \left[\text{"} \hat{p} = -i \frac{\partial}{\partial x} \text{"} \right]$$

This is meant to be a map of fcts.

- Let $|p\rangle$ & $|q\rangle$ be the eigenstates of $\hat{p}$ & $\hat{q}$ with eigenvalues p & q:

$$\hat{p}|p\rangle = p|p\rangle \quad ; \quad \hat{q}|q\rangle = q|q\rangle .$$

- The corresponding fcts. are well-known:

$$|p\rangle = (q' \mapsto e^{ipq'})$$
$$|q\rangle = (q' \longmapsto \delta(q'-q)) .$$

- It is easy to check
$$\langle q | q' \rangle = \delta(q-q')$$
$$\langle p | p' \rangle = (2\pi)\,\delta(p-p')$$
$$\langle p | q \rangle = e^{-ipq},$$

as well as:
$$\mathbb{1} = \int dq\, |q\rangle\langle q| = \int \frac{dp}{2\pi}\, |p\rangle\langle p| \,.$$

[Check all of this!]

- We now want to know the <u>transition amplitude</u> from a state
$$|q_a\rangle \text{ at } t_a = 0 \qquad \text{to} \qquad |q_b\rangle \text{ at } t_b = t.$$

- This amplitude, which by def. reads $\langle q_b | e^{-iHt} | q_a \rangle$, will now be calculated:

$$\langle q_b | e^{-iHt} | q_a \rangle = \langle q_b | \underbrace{e^{-iH\Delta} e^{-iH\Delta} \cdots e^{-iH\Delta}}_{n \text{ factors}} | q_a \rangle \qquad ; \quad \Delta = t/n$$

$$= \prod_{i=1}^{n-1} \left(\int dq_i \right) \langle q_b | e^{-iH\Delta} | q_{n-1} \rangle \langle q_{n-1} | e^{-iH\Delta} | q_{n-2} \rangle \cdots \langle q_1 | e^{-iH\Delta} | q_a \rangle$$

$$\langle q_{i+1} | e^{-iH\Delta} | q_i \rangle = \int \frac{dp}{2\pi} \underbrace{\langle q_{i+1} | p \rangle}_{e^{ipq_{i+1}}} \langle p | \underbrace{e^{-iH\Delta}}_{\approx \mathbb{1} - iH\Delta + O(\Delta^2)} | q_i \rangle$$

(The $O(\Delta^2)$ pieces will be irrelevant for $\Delta \to 0$ & $n \to \infty$.)

Use that:
$$\langle p | \mathbb{1} | q \rangle = \langle p | q \rangle = e^{-ipq}$$
$$\langle p | H | q \rangle = \langle p | \left(\frac{\hat{p}^2}{2m} + V(\hat{q}) \right) | q \rangle$$
$$= \left(\frac{p^2}{2m} + V(q) \right) \langle p | q \rangle = \left(\frac{p^2}{2m} + V(q) \right) e^{-ipq}$$

$$\Rightarrow \langle q_{i+1}|e^{-iH\Delta}|q_i\rangle \simeq \int \frac{dp}{2\pi}\, e^{ip(q_{i+1}-q_i)}\left[1-i\left(\frac{p^2}{2m}+V(q_i)\right)\Delta\right]$$

$$\simeq \int\frac{dp}{2\pi}\exp i\left[p(q_{i+1}-q_i)-\left(\frac{p^2}{2m}+V(q_i)\right)\cdot\Delta\right].$$

- change of int.-variable: $p\to p-(q_{i+1}-q_i)\cdot m/\Delta$
- application of $\int_{-\infty}^{\infty}dx\,e^{-ax^2}=\sqrt{\frac{\pi}{a}}$

$$\Rightarrow \langle q_{i+1}|e^{-iH\Delta}|q_i\rangle \simeq \frac{1}{\sqrt{2\pi i\Delta/m}}\exp i\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2-V(q_i)\right]\cdot\Delta$$

$$\simeq \underbrace{\frac{1}{C(\Delta)}}_{\text{def.}}\cdot\exp i\,\underbrace{L(q,\dot q)}\cdot\Delta$$

This requires thinking of q_i & q_{i+1} as of $q(t=\Delta\cdot i)$ & $q(t=\Delta\cdot(i+1))$

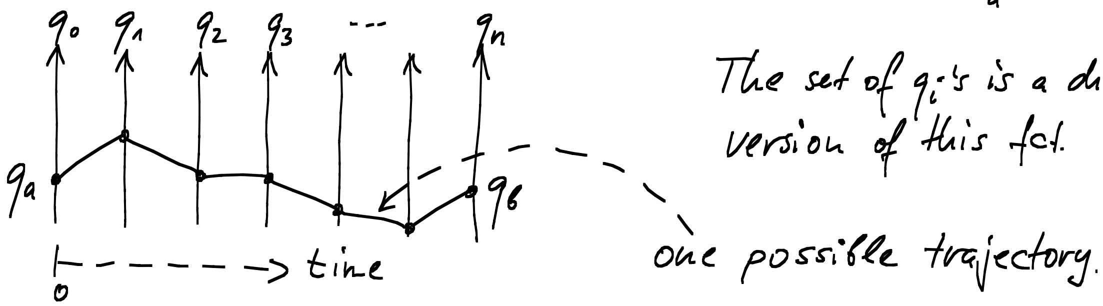
- In total, this gives:

$$\langle q_b|e^{-iHt}|q_a\rangle = \lim_{\Delta\to 0}\frac{1}{C(\Delta)}\prod_{i=1}^{n-1}\left(\int\frac{dq_i}{C(\Delta)}\right)\underbrace{\exp i\sum_{i=0}^{n-1}\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2-V(q_i)\right]\cdot\Delta}_{\exp i\int_{t_a}^{t_b}dt\,L(q(t),\dot q(t))}$$

(with $q_0\equiv q_a$)

The set of q_i's is a discretized version of this fct.

<u>phys. picture:</u>



<u>Or:</u> (less precise but more intuitive)

$$\left\| \; \langle q_b | e^{-iHt} | q_a \rangle = \int \mathcal{D}q \, e^{iS[q]} \; \right\| \qquad \left(\text{where } S[q] = \int_{t_a}^{t_b} dt\, L(q,\dot{q}) \right)$$

$\uparrow$ integral over all functions (= paths, in this case) for which $q(t_a) = q_a$ & $q(t_b) = q_b$.

This is the famous path- (or, more generally, functional-) integral method in QM. (<u>Note</u>: From this, it can be intuitively understood that the <u>class. trajectory</u> dominates. Trajectories whose S differs from $S_{extrem.}$ by more than $O(1)$ (i.e. $O(\hbar)$) are suppressed by the oscill. behaviour of e^{iS}.)

1.2 Correlation fcts. in QM

- Let us write our previous result in the Heisenberg picture:

$$\langle q_b | e^{-iH(t_b - t_a)} | q_a \rangle = \langle q_b | e^{iH(0-t_b)} e^{-iH(0-t_a)} | q_a \rangle$$

$$\equiv \langle q_b, t_b | q_a, t_a \rangle = \int \mathcal{D}q \, e^{iS[q]} \qquad q(t_{a/b}) = q_{a/b}\text{, as above.}$$

- An obvious generalization of the l.h. side is

$$\langle q_b, t_b | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | q_a, t_a \rangle .$$

[Such <u>correlation</u> fcts. (between $\hat{q}$'s at different times) are the QM-analogues of Green's-fcts. $\langle 0 | \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) | 0 \rangle$, which we will need in QFT.]

- Let $m = 1$ (only to save writing) and go to the Schrödinger picture:

$$\langle q_b, t_b | \hat{q}_t | q_a, t_a \rangle = \langle q_b | e^{-iH(t_b - t)} \hat{q} e^{-iH(t-t_a)} | q_a \rangle$$

- let $\hat{q} = \mathbb{1} \cdot \hat{q} = \int dq |q\rangle\langle q| \hat{q} = \int dq\, |q\rangle q \langle q|$

$$\Rightarrow \langle \cdot\cdot \rangle = \int dq \underbrace{\langle q_b, t_b | q, t \rangle}\, q \underbrace{\langle q, t | q_a, t_a \rangle}$$

assuming $t_b > t > t_a$, apply twice our fund. int. expression (in the discrete form). The two products

$\left(\prod_{i=1}^{n_1 - 1} \int dq_i \right)$ & $\left(\prod_{i=1}^{n_2 - 1} \int dq_i \right)$ can be relabelled and combined with $\int dq$ to give a single path int. formula. We give directly the continuum form:

$$\langle q_b, t_b | \hat{q}_t | q_a, t_a \rangle = \int \mathcal{D}q\, q(t)\, e^{iS[q]}.$$

- For $t_b > t_m > \cdots > t_1 > t_a$, this generalizes to

$$\langle q_b, t_b | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | q_a, t_a \rangle = \int \mathcal{D}q\, q(t_m) \cdots q(t_1)\, e^{iS[q]}.$$

- What we will really need are vacuum corr. fcts:

$$\langle 0, t = \infty | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | 0, t = -\infty \rangle.$$

- For this purpose, consider first (let again $m = 1$)

$$\langle q', T | \hat{q}_t | q, -T \rangle \qquad (\text{for } T > t > -T)$$

$$= \langle q' | e^{-iH(T-t)} \hat{q} e^{-iH(t-(-T))} | q \rangle$$

- Let $H \to H' = (1 - i\varepsilon) \cdot H$

(we will let $\varepsilon \to 0$ at the end).

- Use the fact that, generically, $|q\rangle = \alpha|0\rangle + \cdots$ orthog. states

$$\Rightarrow \langle q'|e^{-iH'(T-t)}\hat{q}\,e^{-iH'(t+T)}|q\rangle$$

$$\sim \langle 0|e^{-iH'(T-t)}\hat{q}\,e^{-iH'(t+T)}|0\rangle$$

at large T, since $e^{-\varepsilon H}$ suppresses the vacuum the least.

- We now take the limit $T\to\infty$. We also observe that $|q\rangle$ & $|q'\rangle$ are irrelevant (since only their projection on the vacuum matters; the size of this vac. contribution is important for the normalization, but we can't keep track of the normalization anyway).

$$\Rightarrow \quad \langle 0|\hat{q}_t|0\rangle \sim \int Dq\, q(t)\, e^{iS'[q]}$$

any fct. $q(t)$ (no boundary condition)

or, more generally,

$$\langle 0|\hat{q}_{t_m}\cdots\hat{q}_{t_1}|0\rangle \sim \int Dq\, q(t_1)\cdots q(t_m)\, e^{iS'[q]}.$$

(This follows by repeating our previous argument for more than one $\hat{q}$, with $t_1 < t_2 < \cdots < t_m$.)

- Equivalently, we can write

$$\langle 0|T\,\hat{q}_{t_1}\cdots\hat{q}_{t_m}|0\rangle \sim \int Dq\, q(t_1)\cdots q(t_m)\, e^{iS'[q]}$$

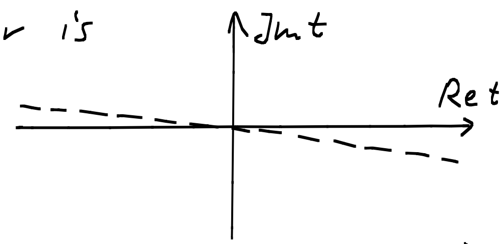
for <u>any</u> set of times $t_1 \cdots t_m$.

- We now return to the "$i\varepsilon$" contained in $H' = (1-i\varepsilon)H$:

Since H' always appears in the form $\int H' dt$,

we may as well write $\int H(1-i\varepsilon)dt = \int H dt'$,

keeping in mind that the integr. contour is



> Note: We can also perform a full rotation of the integration contour by $90°$ (Wick rotation), such that the time-evolution operator is $e^{-H\tau}$ (with τ real). The corresp. path integral is called <u>euclidean</u>.

- Thus, we will from now on simply write

$$\langle 0 | T \hat{q}_{t_1} \cdots \hat{q}_{t_m} | 0 \rangle \sim \int Dq \, q(t_1) \cdots q(t_m) \, e^{iS[q]}$$

with small neg. imag. part in time-int.

- This imaginary part has a clear intuitive meaning:

At large, negative S (corresponding to large V in $S = T - V$, i.e. to being far away from the minimum or the classical trajectory), the e^{iS}-factor oscillates rapidly (with changing $q(t)$). To make the integral convergent, we give t or S a small imaginary part by multiplying with $(1-i\varepsilon)$. In fact, it will turn out to be sufficient to give this part to the leading potential term in $V(q)$:

$$V(q) = \frac{m^2}{2} q^2 + \cdots \quad \longrightarrow \quad V(q) = \frac{m^2 - i\varepsilon}{2} q^2 + \cdots$$

$\Rightarrow$ We will now simply write $e^{iS[q]}$ and integrate over real t, but remember to use $m^2 - i\varepsilon$ instead of m^2 whenever necessary for convergence (and take $\varepsilon \to 0$ in the end). In QFT (see below) this will reproduce the $i\varepsilon$-prescription of the Feynman propagator introduced earlier.

- Finally, we deal with the normalization by always focussing on

$$\frac{\langle 0|T\,\hat{q}_{t_1}\cdots\hat{q}_{t_m}|0\rangle}{\langle 0|0\rangle} = \frac{\int \mathcal{D}q\, q(t_1)\cdots q(t_m)\, e^{iS[q]}}{\int \mathcal{D}q\, e^{iS[q]}}$$

This is now an equality since the normalization problem that appeared in $|q\rangle = \alpha|0\rangle + \cdots \longrightarrow |0\rangle$ and the limit $T\to\infty$ (as well as all issues with $1/C(\Delta)$-prefactors) cancel out.

<u>Specific literature:</u> (path- or funct. integrals)
- Rivers
- Zinn-Justin
- Schulman

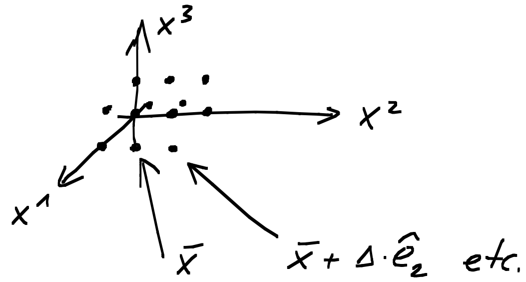
1.3 Functional integral for the scalar field

- recall: $S = \int d^4x\, \mathcal{L} = \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\right)$

  also: $S = \int dt\, L$ with $L = \int d^3x \left(\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\right)$.

$$L = \int d^3x\left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - V(\varphi)\right) \xrightarrow[\substack{\pi = \dot{\varphi} \\ \text{(Leg.-trf.)}}]{} H = \int d^3x\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + V(\varphi)\right)$$

- Put this on a (spatial) lattice:



$$\Downarrow$$

$$H = \sum_{\bar{x}} \left(\frac{1}{2}\pi(\bar{x})^2 + \frac{1}{2}(\bar{\nabla}\varphi)^2_{\bar{x}} + V(\varphi(\bar{x})) \right)$$

where $$(\bar{\nabla}\varphi)_{\bar{x},\alpha} = \frac{1}{\Delta}\left(\varphi(\bar{x}+\Delta\hat{e}_\alpha) - \varphi(\bar{x}) \right)$$

- Interpreting this as a (q.m.) many-particle system with coordinates $\varphi_{\bar{x}}$ and conj. momenta $\pi_{\bar{x}}$, we can write

$$H = \frac{1}{2}\sum_{\bar{x}} \pi(\bar{x})^2 + \tilde{V}\left(\{\varphi(\bar{x}),\ \bar{x}\in \text{lattice}\} \right).$$

$$\uparrow$$

$\sum(\bar{\nabla}\varphi)^2_{\bar{x}}$ has been absorbed here.

- This is just a set of (q.m.) particles interacting via the potential (the kin. term is canonical). We can apply our path-int. derivation without change (just with more variables), e.g.

$$|q\rangle \longrightarrow |q_1\cdots q_n\rangle \longrightarrow |\{\varphi(\bar{x}),\ \bar{x}\in \text{latt.}\}\rangle$$

(many part.) (field theory)

short form

$$\langle q'|q\rangle = \delta(q'-q) \longrightarrow \cdots \longrightarrow \langle\varphi'|\varphi\rangle = \prod_{\bar{x}} \delta(\varphi(\bar{x}) - \varphi'(\bar{x}))$$

etc.

- The result is: (Check this!)

$$\langle\varphi_b| e^{-iH(t_b-t_a)} |\varphi_a\rangle = \prod_{\bar{x}}\left(\int \mathcal{D}\varphi(\bar{x})\right) \exp i \sum_i \left\{ \sum_{\bar{x}} \left[\frac{1}{2}\left(\frac{\varphi_{i+1}(\bar{x}) - \varphi_i(\bar{x})}{\Delta_t} \right)^2 \right] - \tilde{V}\left(\{\varphi_i(\bar{x}),\ \bar{x}\in\text{latt.}\}\right) \right\}\cdot \Delta_t$$

<u>as before:</u>

$$\int\mathcal{D}\varphi(\bar{x}) = \lim_{\Delta_t\to 0} \frac{1}{C(\Delta_t)} \prod_i \int \frac{d\varphi_i(\bar{x})}{C(\Delta_t)}$$

As before, we view the $\varphi_i(\bar{x})$ (for each $\bar{x}$) as the discr. version of a fct. $\varphi(t,\bar{x})$. At the same time, we can return to the continuum treatment of $\bar{x}$:

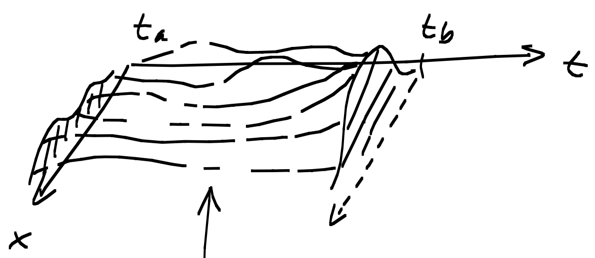
$$\langle \varphi_b | e^{-iH(t_b-t_a)} | \varphi_a \rangle = \int \mathcal{D}\varphi \, e^{iS[\varphi]} \quad \text{where}$$

$$S[\varphi] = \int_{t_a}^{t_b} dt \, L\underset{\substack{\uparrow \\ \varphi(t,\bar{x}) \text{ for all } \bar{x}}}{[\varphi(t)]} = \int_{t_a}^{t_b} dt \int d^3\bar{x} \, \mathcal{L}(\varphi)$$

Here $\int \mathcal{D}\varphi$ is the "summation" over all fcts. $\varphi(x) = \varphi(t,\bar{x})$ with $\varphi(t_a,\bar{x}) = \varphi_a(\bar{x})$ & $\varphi(t_b,\bar{x}) = \varphi_b(\bar{x})$.

(Check all of this, unless obvious.)

<u>phys. picture:</u>



fct $\varphi(t,x)$ interpolating between $\varphi_a(x) = \varphi(t_a,x)$ & $\varphi_b(x) = \varphi(t_b,x)$.

<u>Note:</u> All of the above can be done (at least formally) without discretizing $\bar{x}$ and without the restriction to initial/final states with fixed field-values. This requires the concept of the "Schrödinger wave functional". To understand this, let us go, step by step, from QM to many-body-QM to QFT:

<u>QM:</u> state: $\psi(q)$; example: $\psi(q) = \delta(q-q^0)$ (i.e. the particle position is known)

<u>many-Body-QM:</u> state: $\psi(q_1, \cdots, q_n)$; example: $\psi(q_1 \cdots q_n) = \prod_{k=1}^{n} \delta(q-q^0)$

(i.e. all particle positions are known)

<u>QFT:</u> state: $\Psi[\varphi]$ – This "wave functional" associates with every function $\varphi: \bar{x} \mapsto \varphi(\bar{x})$ a complex number $\Psi[\varphi]$ and hence a probability $|\Psi[\varphi]|$ to measure this particular set of field values (i.e. this $\varphi(\bar{x})$ for every $\bar{x}$).

example: $\Psi[\varphi] = \delta[\varphi - \varphi^0] \equiv \prod_{\bar{x}} \delta(\varphi(\bar{x}) - \varphi^0(\bar{x}))$

formal product over all $\bar{x}$

A more practically useful realization of this δ-fct.-example arises if we replace the integral over all fcts. by an integral over all corresponding Fourier coefficients:

$$\int Dq\, e^{iS[q]} \sim \prod_i \left(\int d\lambda_i\right) e^{iS[q]}$$

e.g. $q(t_a) = 0$
$q(t_b) = 0$
(defined by discretizing t as above)

with $q(t) = \sum_i \lambda_i q_i(t)$

orthonormal basis of fcts. (e.g. Fourier modes) respecting the boundary conditions, in our case $q_i(t_a) = q_i(t_b) = 0$.

This is a statement (without proof) about the functional dependence on the parameters of S, i.e. ignoring normalization issues.

Now our δ-fct. example reads: $\Psi[\varphi] = \prod_i \delta(\lambda_i - \lambda_i^0)$

Finally, again in complete analogy with QM, we have:

$$\langle 0 | T\varphi(x_1)\cdots\varphi(x_m)|0\rangle = \frac{\int D\varphi\, \varphi(x_1)\cdots\varphi(x_m)\, e^{iS}}{\int D\varphi\, e^{iS}}$$

to be divided by $\langle 0|0\rangle$, unless $=1$.

Heisenberg-picture operators following from $\varphi(\bar{x}) \equiv \varphi(t=0,\bar{x})$ with the <u>full interacting dynamics</u> of the theory.

Here $S = \int d^4x\, \mathcal{L}$, $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$

& $V(\varphi) = \frac{1}{2}(m^2 - i\varepsilon)\varphi^2 + \cdots$ higher powers

leads to a term $e^{-\frac{1}{2}\varepsilon\varphi^2}$ which suppresses non-vac. contributions at $t_{a/b} \to -/+\infty$. One can view it simply as a convergence factor.

> This page can be taken as a starting point for the development of QFT. The only extra information you need from QFT I is the LSZ-formula, which relates time-ordered correlation fcts. to scattering amplitudes and thus tells us <u>why</u> we should be interested in this particular type of correlation functions.