10 Generating Functionals, Effective Actions, Spont. Symm. Breaking, Condensed Matter, AdS/CFT

(This is only a brief overview – the above topics deserve a much more detailed discussion, but there is no time left.)

10.1 Gen. Fct. & Eff. Act.

Recall that $Z(j) = \int D\varphi \, e^{iS[\varphi] + ij\varphi}$

$$ij\varphi = i\int d^4x\, j(x)\varphi(x)$$

is the generating functional for (time-ordered) Green's fcts. We can give it a more physical interpretation by recalling that

$$Z(j) = \langle 0| e^{-iHt} |0\rangle \Big|_{t\to\infty}.$$

Let us compare this to the partition fct. familiar from thermodynamics:

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_i \langle i| e^{-\beta H} |i\rangle .$$

The r.h. side can be rewritten as a functional integral (as we did before, but with (Wick-rotated) euclidean time:

$$Z = \int D\varphi \, e^{-\int_0^\beta d\tau\, d^3x\, \mathcal{L}_E(\varphi)}$$

$\int D\varphi$: integral over field configs. periodic in τ with period β, to account for $\sum_i \langle i|\cdots|i\rangle$.

Thus, we can think of Z as the partition fct. of our theory at $T=0$ ($\beta = \infty$) in the presence of an external source $j(x)$.

<u>Note:</u>
- The Wick rotation becomes irrelevant at $\beta \to \infty$.
- For fermions, we have to impose antiperiodicity rather than periodicity in τ.
- We can think of $j \neq 0$ as of an external magnetic field applied to a thermodyn. system. $\Rightarrow Z[j]$ is the part. fct. at $T=0$ & $j \neq 0$.

- We now define another gen. functional,

$$W[j] = -\ln Z[j],$$

the thermodynamic analogue of which is that of the Helmholtz free energy $F(B) = -T \ln Z(B)$ with $B \hat{=} j$ the ext. magn. field.
(We now always work with euclidean metric, also in QFT $T=0$.)

- We know that $F = E - TS$, i.e. $F = \langle \hat{H} \rangle$ at $T=0$.
- Explicitly: $$Z = e^{-W} = \int D\varphi \, e^{-S_E} = \langle 0 | e^{-\beta \hat{H}} | 0 \rangle = \exp(-\beta \langle \hat{H} \rangle)$$

since $|0\rangle$ is eigenstate of $\hat{H}$

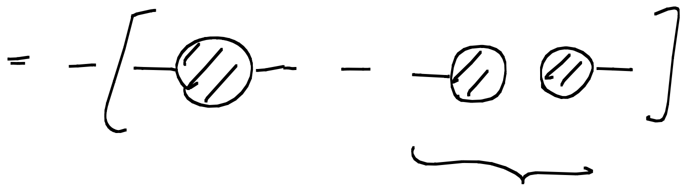
$\Rightarrow \quad W = \beta \langle \hat{H} \rangle = \beta V \varrho_E = (t \cdot V)\varrho_E = (\mathrm{Vol}_4) \cdot \varrho_E$

$\Rightarrow \quad W/\mathrm{Vol}_4$ is the energy density of our system in presence of a source j.

- $W[j]$ generates the <u>connected</u> Green's fcts., e.g.,

$$\frac{\delta^2 W}{\delta j_1 \delta j_2} = -\frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2} \ln Z = -\frac{\delta}{\delta j_1}\left(\frac{1}{Z}\frac{\delta}{\delta j_2} Z\right)$$

$$= -\frac{1}{Z}\frac{\delta^2}{\delta j_1 \delta j_2} Z + \left(\frac{1}{Z}\frac{\delta Z}{\delta j_1}\right)\left(\frac{1}{Z}\frac{\delta Z}{\delta j_2}\right)$$



this subtracts the disconnected part.

(This argument can be extended to all n-point fcts.
$\frac{\delta^n}{\delta j_1 \cdots \delta j_n} W$. See e.g. book by Rivers.)

- An alternative argument assumes that $(-W)$ generates conn. Green's fcts. and shows that $e^{-W} = 1 + (-W) + \frac{1}{2}(-W)^2 + \cdots$ generates all.
($\rightarrow$ Weinberg)

- Given the interpretation of $W[j]$ as the (Helmholtz) free energy (density) in the presence of an ext. source j (i.e. as a thermodyn.-potential), it is natural to define another thermodyn. potential (the Gibbs free energy) via a Legendre trf.:

$$\varphi[j] \equiv -\frac{\delta W[j]}{\delta j}$$

$$\underbrace{\Gamma[\varphi]}_{\equiv \text{"effective action"}} = -W[j[\varphi]] - \underbrace{j[\varphi]\cdot\varphi}_{\int d^4x\, j(x)\varphi(x),\ \text{as usual.}}$$

- Obviously, $\varphi[j] = -\frac{1}{Z}\frac{\delta}{\delta j}\int D\varphi'\, e^{-S[\varphi']-j\varphi'} = \langle\varphi'\rangle$,
i.e. $\varphi[j]$ is the VEV of the quantum field in presence of j.

- As expected for a Leg. trf.,

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} = -\underbrace{\frac{\delta W}{\delta j}}_{\varphi}\cdot\frac{\delta j}{\delta\varphi} - \frac{\delta j}{\delta\varphi}\cdot\varphi - j[\varphi] = -j[\varphi],$$

i.e., at stationary points of $\Gamma[\varphi]$, $j[\varphi]=0$ and

$$\Gamma[\varphi] = -W[j[\varphi]] = -VT\rho_E\,.$$

For $\varphi = \text{const.}$, we define $V_{\text{eff.}}(\varphi) = -\frac{1}{VT}\Gamma[\varphi = \text{const.}]$.

Minima of $\Gamma[\varphi]$ with $\varphi = \text{const.}$ correspond to minima of $\underbrace{V_{\text{eff.}}}_{\equiv\text{"eff. potential"}}$ correspond to Poinc.-inv. vacua of the theory.

- To interpret $\Gamma[\varphi]$ as a generating functional, observe that

$$\frac{\delta\Gamma}{\delta\varphi} = -j \;\Rightarrow\; -\frac{\delta}{\delta j}\frac{\delta\Gamma}{\delta\varphi} = \mathbb{1} \;\Rightarrow\; -\frac{\delta\varphi_1}{\delta j_3}\frac{\delta^2\Gamma}{\delta\varphi_1\delta\varphi_2} = \mathbb{1}_{32}$$

$$\Rightarrow\; \frac{\delta^2 W}{\delta j_3\,\delta j_1}\cdot\frac{\delta^2\Gamma}{\delta\varphi_1\delta\varphi_2} = \mathbb{1}_{32} \;\Rightarrow\; \frac{\delta^2\Gamma}{\delta\varphi_1\delta\varphi_2} = \left(\frac{\delta W}{\delta j_1\,\delta j_2}\right)^{-1} \text{"="} \left(p^2+m^2+\Pi(p^2)\right)$$

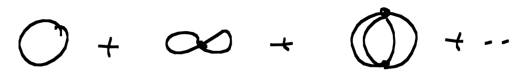
$$\Rightarrow \quad \frac{\delta^2 \Gamma}{\delta\varphi_1 \delta\varphi_2} = \frac{\delta^2 S_{free}}{\delta\varphi_1 \delta\varphi_2} + -\otimes-$$

$\uparrow$ 1PI – two-point-fcts.

That's also an easy way to read off the Feynm. rules for Γ (and hence for W & Z)

- This extends to all n-point-fcts.: <u>Γ generates 1PI n-point-fcts.</u>

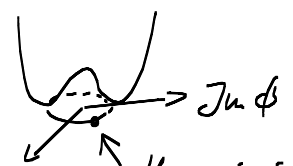
[In particular, $V_{eff} = V_{free}$ + 1PI vacuum diagrams, i.e.



$$\sim \int d^4k_E \ln(k^2+m^2) \Big]$$

10.2 Spontaneous symm. breaking

It can happen that $S_{class.}$ and Γ have a symm., but the minimum of Γ is not invar. under this symm. In the simplest case, this happens at tree-level: $\mathcal{L} = |\partial\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4$ with $m^2 < 0$.



In QM-language: $|0\rangle$ is not invar. under a symm. of $\hat{H}$.
In this case, there is always (at least one) massless scalar χ corresponding to changing ϕ_{min} "along the S^1". (roughly speaking, $\chi \equiv \arg\phi$.). χ is called the "Goldstone boson" and the above argument can be promoted to the "Goldstone theorem".

[This can not happen in $d \leq 2$ at any $T \neq 0$, roughly speaking since fluctuations of χ are so strong that the vacuum configurations permanently "explore" the whole S^1. $\Rightarrow$ "Mermin–Wagner–theorem".]

- If the symm. is gauged, the gauge boson acquires a mass (thereby getting 3 rather than 2 d.o.f.). Thus, the Goldstone boson is "eaten" by the vector boson. No massless particle is left.
- More explicitly:

$$|D\phi|^2 = D_\mu\phi(D^\mu\phi)^* = \partial_\mu\phi\,\partial^\mu\phi^* + iA_\mu\phi\,\partial^\mu\phi^*$$
$$- iA_\mu\phi^*\partial^\mu\phi - \underbrace{A_\mu A^\mu|\phi|^2}$$

mass for A_μ
for $\langle\phi\rangle \neq 0$.

- Furthermore: $\phi = \rho e^{i\chi}$

$$\partial^\mu\phi = (\partial^\mu\rho)e^{i\chi} + i(\partial^\mu\chi)\,\phi$$

$$\Rightarrow\ iA_\mu\left[\phi\,\partial^\mu\phi^* - (\partial^\mu\phi)\,\phi^*\right] = 2A_\mu\,\partial^\mu\chi\,|\phi|^2$$

- We see that, while χ has <u>no mass</u> term, it mixes (via a kinetic or derivative term) with A_μ, which <u>has a mass term</u>. This explains at an intuitive level why χ disappears from the massless spectrum.

- Technically, one can chose a ("unitary") gauge where $\chi = 0$ or a "covariant" *) gauge where χ remains relevant. In the former, only phys. d.o.f. propagate and unitarity is manifest. In the latter, the vector propagator is similar to what we discussed before and renormalizability (in particular no power-divergences) is manifest. Going back and forth between the two requires a gauge-inv. regularization (dim.reg.). This establishes the consistency of spont. broken gauge-theories as renorm. QFTs. ('t Hooft, Veltman).

*) An important and widely used example is the R_ξ-gauge:

Let $\phi = \frac{1}{\sqrt{2}}\left((v + h(x)) + i\varphi(x)\right)$ [at leading order h, φ corresp. to $\rho - v$, χ]

Choose gauge fixing function

$$G = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu - \xi v \varphi\right).$$

$\Rightarrow$ The mixing term between χ & (∂A) [$\equiv$ between φ & (∂A)] discussed above is cancelled by the corresponding mixing term from $\mathcal{L}_{g.f.} \sim G^2$.

10.3 QFT in CMT (→ Altland / Simons; old classic: Fetter / Walecka)

- Consider a QM many-particle system
- Each particle can be in discrete set of states labelled by k (think of discrete momenta of particles in a box)
- Generic state: $|\Psi\rangle = |n_{k_1} n_{k_2} \cdots\rangle$

  (i.e. n_{k_1} particles in state k_1, etc.)
- Fock space description:

$$|\Psi\rangle = \prod_k \frac{(a_k^+)^{n_k}}{\sqrt{n_k!}}|0\rangle \quad ; \quad \text{"}[a_k, a_q^+\}\text{"} = \delta_{kq}$$

(Bosons or fermions)

- Generic one- / two-particle operators:

$$\hat{O}_1 = \sum_{kq} \langle k|\underset{\sim}{\hat{O}}_1|q\rangle\, a_k^+ a_q$$

$$\hat{O}_2 = \sum_{kk'qq'} \langle kk'|\underset{\sim}{\hat{O}}_2|qq'\rangle\, a_k^+ a_{k'}^+ a_q a_{q'}$$

(Examples: $\underset{\sim}{\hat{O}}_1 = V(\hat{r})$; $\underset{\sim}{\hat{O}}_2 = V(|\hat{r} - \hat{r}'|)$)

- Demonstration: $\langle 0| a_\ell \hat{O}_1 a_p^+ |0\rangle = \cdots = \langle \ell | \underset{\sim}{O}_1 | p\rangle$ etc.
- let $a(x) = \sum_k \langle x | k \rangle a_k$ with $|x\rangle$ being position eigenstates.
- Use the notation $x \to \bar{x}$ since x is, of course, a vector in $d>1$.

$$\Rightarrow \quad \hat{H} = \int d^dx \, a^+(\bar{x}) \left[\frac{\hat{p}^2}{2m} + V(\bar{x}) \right] a(\bar{x})$$

$$+ \frac{1}{2} \int d^dx \, d^dx' \, V(|\bar{x} - \bar{x}'|) \, a^+(\bar{x}) a^+(\bar{x}') a(\bar{x}) a(\bar{x}')$$

where $\hat{p} = -i\partial_{\bar{x}}$

and we used the fact that $a^+(x)|0\rangle = |x\rangle$.

(Demonstration: show that $\langle k | a^+(x) | 0\rangle = \langle k | x\rangle$ using the definition of $a(x)$ given above.)

This is a rather general (normal ordered) QFT-Hamiltonian useful in CMT.

- It is easy to go to functional-int. repres. for observable quantities of interest, e.g. Z. We use coherent states:

$$|\varphi\rangle = e^{a_i^+ \varphi_i} |0\rangle$$

(φ bosonic or fermionic; i – set of all indices, including e.g. $\bar{x}$ or $\bar{k}$)

$$a_i |\varphi\rangle = \varphi_i |\varphi\rangle$$

$$Z = \sum_n \langle n | e^{-\beta(\hat{H} - \mu \hat{N})} | n \rangle$$

Apply: $\mathbb{1} = \int D\bar{\varphi} \, D\varphi \, e^{-\sum_i \bar{\varphi}_i \varphi_i} |\varphi\rangle\langle\bar{\varphi}|$

(after splitting $e^{-(\cdots)}$ in many factors; before & after each factor)

Following precisely the analysis of Sect. 3.2 & its fermionic counterpart, we find:

$$Z = \int D\bar{\psi}\, D\psi\, e^{-S} \quad ; \quad S = \int_0^\beta d\tau \left[\bar{\psi}\,\partial_\tau \psi + H(\bar{\psi},\psi) - \mu N(\bar{\psi},\psi)\right]$$

period. or antiper. in τ

$\psi, \bar{\psi}$ "class." versions of a, a^+.

$$\Downarrow$$

$$S = \int_0^\beta d\tau \Big[\underbrace{\bar{\psi}\,\partial_\tau \psi + \frac{1}{2m}\bar{\psi}(-\bar{\nabla}^2)\psi}_{\bar{\psi}\left(\dot{\psi} - \frac{\bar{\nabla}^2}{2m}\psi\right)} + H_{int.} - \mu N \Big]$$

$\Rightarrow$ NR QFT (for fermions or bosons) with Schröd. eq. at leading order (instead of Klein-Gordon or Dirac eq.)

$$-\frac{d}{d\tau}\psi = -\frac{\bar{\nabla}^2}{2m}\psi \quad \xrightarrow[\tau \to it]{} \quad i\partial_t \psi = -\frac{\bar{\nabla}^2}{2m}\psi$$

$$\partial_t \psi = -i\left(-\frac{\bar{\nabla}^2}{2m}\right)\psi \quad \checkmark$$

(This, of course, also follows as the NR-limit of our previous rel. QFTs for fermions & bosons, including the spin index for fermions which was suppressed above.
Cf. the books of Zinn-Justin (Sect. 7.6) and of Zee (Sect. III.5) for more details on the NR-limit of a rel. QFT.)

10.4 AdS/CFT, Renormalization Group

AdS/CFT: Correspondence between gravity in Anti-de-Sitter space in $d+1$ dims. and Conf. Field Theory in d dims.

Motivation: Black hole: entropy $\sim A_{horizon}$ $\Rightarrow$ the d.o.f. of a (d+1)-dim. system can be viewed as being localized on a surface (d-dims.). This <u>holographic principle</u> is conjectured to hold in a much wider context. It is best illustrated starting from a CFT.

- To define a CFT, let us step back and talk, once again, about renormalization. Recall the (obvious) diff. equ. for a dim.less observable R:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) R(g, Q/\mu) = 0.$$

This generalizes to the "Callen-Symanzik-Equ." for Greens fcts.:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma(g)\right) G^{(n)}(\{x_i\}, g, \mu) = 0.$$

One can think of μ as the cutoff and of the set of theories (parameterized by μ) arising from this diff.eq. as the result of symm. trfs. which correspond to lowering μ (renormalization group). This corresponds to "integrating out" degrees of freedom between μ_1 and μ_2 ($\mu_1 > \mu_2$). (This is the "Wilsonian" point of view.)

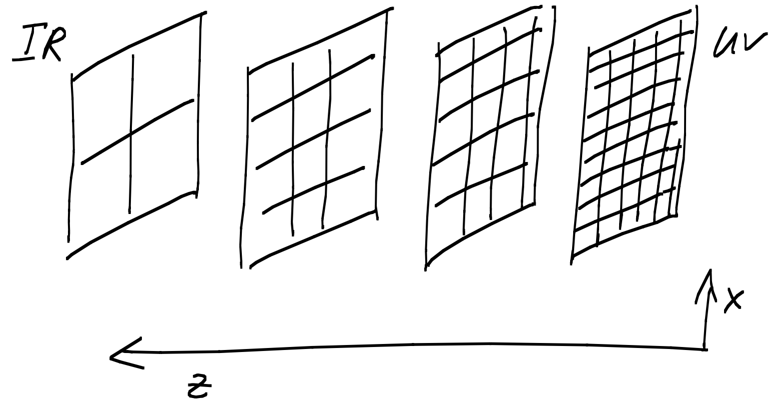
- If $\beta \equiv 0$, all these theories are "self-similar" and we are dealing with a CFT.

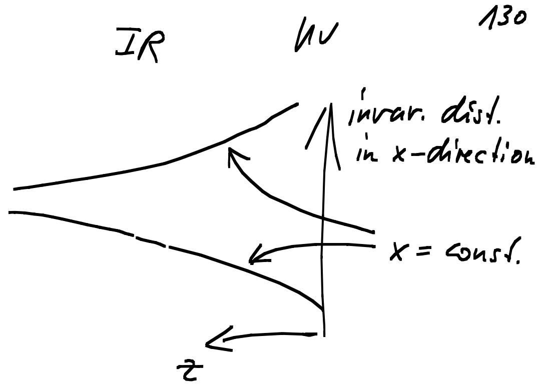
[We now follow McGreevy, 0909.0518]

- Changing μ is equivalent to rescaling coordinates $x^\mu \to \lambda x^\mu$ ($\mu = 0, \dots, d-1$). To find an equivalent (d+1)-dim. gravit. system, introduce an extra coordinate z and look for a metric which solves Einstein's eqs. and is inv. under $z \to \lambda z$:

$$ds^2 = \left(\frac{L}{z}\right)^2 \left(\eta_{\mu\nu}\, dx^\mu dx^\nu + dz^2\right)$$

This is AdS space with AdS-radius L.





It extremizes the action $S = \frac{1}{16\pi G_N}\int d^{d+1}x\,\sqrt{-g}\,(\mathcal{R} - 2\Lambda)$;

$$-2\Lambda = \frac{d(d-1)}{L^2}.$$

(The lines in each slice of the picture are meant to be separated by the same invariant distance. At larger z, this corresponds to a larger separation in the coordinate x.)

- The basic physical idea is that the different slices of AdS above correspond to modes of different wavelength in the CFT, which we imagine to be at the boundary of AdS at $z = 0$. Hence, the CFT is a "hologram" of AdS.

- More technically, our CFT has operators O_1, O_2 etc. and their correl. fcts. $\langle O_1(x_1)\, O_2(x_2) \cdots \rangle$ are generated by $Z = Z(J_1, J_2, \dots)$. We write

$$Z[J] = \left\langle \exp\left[-S - \int J_i O_i\right] \right\rangle_{CFT} = \langle e^{-S_J} \rangle_{CFT}.$$

$J \neq 0$ encodes a perturbation of the bare Lagrangian (corresponding to the UV, i.e. $z = 0$). Hence

dim. of O_i

$$Z[J] = \langle e^{-S_J} \rangle_{CFT} = Z_{\substack{\text{quant.}\\ \text{grav.}}}\Big[\underbrace{\phi(x,z) \approx z^{d-\Delta} J(x)}_{\text{at } z \to 0}\Big] \approx e^{-S_{\text{grav.}}}\Big|_{[\cdots]}.$$

e.g. metric

Thank you very much for your attention and for your questions and comments!

I apologize for all the interesting issues that I did not manage to cover in this course.

Please keep reading the books!