2 Feynman rules in the path integral approach

2.1 The generating functional

Preliminary remark: Functional differentiation

- Let F be a functional of j: $F : \underset{\substack{\uparrow \\ \text{some fct. } x \mapsto j(x)}}{j} \mapsto \underset{\substack{\nwarrow \\ \text{number}}}{F[j]}$

- $\dfrac{\delta F[j]}{\delta j(x)}$ is a number defined by:

$$F[j+\delta j] - F[j] = \int d^4x \, \frac{\delta F[j]}{\delta j(x)} \cdot \delta j(x) + O(\delta j^2).$$

- Examples:

  1) $F[j] = \int d^4y \, j(y) \varphi(y)$; $\dfrac{\delta}{\delta j(x)} \int d^4y \, j(y) \varphi(y) = \varphi(x)$

  2) $F[j] = j(y)$; $\dfrac{\delta}{\delta j(x)} j(y) = \delta^4(x-y)$

- Useful analogies (continuous x $\longrightarrow$ some discrete set $\{x^i\}$)

  1) $\dfrac{\partial}{\partial x^i} \sum\limits_j x^j k^j = k^i$

  2) $\dfrac{\partial}{\partial x^i} x^j = \delta^{ij}$

- Definition of the generating functional:

$$Z[j] = \int D\varphi \, \exp i \int d^4x \left(\mathcal{L}(\varphi, \partial_\mu \varphi) + j(x) \cdot \varphi(x) \right)$$

- Crucial Fact:

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle = \frac{1}{Z[0]} \left(\frac{\delta}{i \delta j(x_1)} \right) \cdots \left(\frac{\delta}{i \delta j(x_n)} \right) Z[j] \Big|_{j=0}$$

(with $\langle 0 | 0 \rangle = 1$)

- <u>Demonstration:</u>

$$\frac{\delta}{\delta j(x)} \int \mathcal{D}\varphi \exp i \int d^4y \left(\mathcal{L} + j(y)\varphi(y)\right) = \int \mathcal{D}\varphi \, i\varphi(x) \exp i \int d^4y \left(\mathcal{L} + j(y)\varphi(y)\right)$$

Let $x = x_1$ and repeat the calculation for $x = x_2, \cdots, x = x_n$

$\Rightarrow$ factor $i\varphi(x_1) \cdots i\varphi(x_n)$ in path integral.

Use also: $Z[0] = \int \mathcal{D}\varphi \, e^{iS}$; the result follows. $\square$

2.2 The free-field case:

$$\mathcal{L}_0 = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2 - i\varepsilon}{2}\varphi^2 \quad ; \quad Z_0[j] = \int \mathcal{D}\varphi \exp i \int d^4x (\mathcal{L}_0 + j\varphi)$$

- Rewrite $S_0 = \int d^4x \, \mathcal{L}_0$:

$$\begin{aligned} S_0 &= \frac{1}{2} \int d^4x \, \varphi(x) \left(-\Box - m^2 + i\varepsilon\right) \varphi(x) \\ &= \frac{1}{2} \int d^4x \int d^4y \, \varphi(x) \, \delta^4(x-y) \left(-\Box_y - m^2 + i\varepsilon\right) \varphi(y) \\ &= \frac{1}{2} \int d^4x \int d^4y \, \varphi(x) \underbrace{\left[\left(-\Box_x - m^2 + i\varepsilon\right) \delta^4(x-y)\right]}_{\equiv \, iD_F^{-1}(x-y)} \varphi(y) \end{aligned}$$

$x \leftrightarrow y$

(Check that D_F^{-1} is indeed the inverse of the Feynman prop.,
i.e. $\int d^4y \, D_F^{-1}(x-y) \, D_F(y-z) = \delta^4(x-z)$.)

- A convenient shorthand for this type of expressions is:

$$i \int d^4x \, \mathcal{L}_0 = \underbrace{-\frac{1}{2} \varphi_x \left(D_F^{-1}\right)_{xy} \varphi_y}_{\text{"summation" over } x,y \text{ implicit}} = \underbrace{-\frac{1}{2} \varphi \, D_F^{-1} \varphi}_{\text{in analogy to matrix-notation}}$$

$\sum_{ij} a_i M_{ij} b_j = a^T M b$

- Hence: $\quad Z_0[j] = \int \mathcal{D}\varphi \, \exp\left(-\frac{1}{2}\varphi^T D^{-1}\varphi + i j \varphi\right)$

$D_F \to D$ for simplicity.

- Change int. variable: $\quad \varphi \to \underbrace{\varphi + iDj}$

$\left(\text{i.e. } \varphi(x) + i\int d^4y \, D(x-y) j(y)\right)$.

$$\Rightarrow Z_0[j] = \int \mathcal{D}\varphi \, \exp\Big[-\frac{1}{2}\varphi^T D^{-1}\varphi - \frac{i}{2} j^T D^T D^{-1}\varphi - \frac{i}{2}\varphi^T D^{-1} D j$$
$$+ \frac{1}{2} j^T D^T D^{-1} D j + i j^T \varphi - j^T D j\Big]$$

- Note: $D^T = D$. (Check this!)

$$\Rightarrow Z_0[j] = \int \mathcal{D}\varphi \left[-\frac{1}{2}\varphi^T D^{-1}\varphi - \frac{1}{2} j^T D j\right] = \underline{\underline{Z_0[0]\exp\left[-\frac{1}{2} j^T D j\right]}}$$

- Now we see also from a conceptual perspective (without knowing the explicit form) that D is the Feynman propagator:

$$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle = \langle T\varphi_1\varphi_2\rangle = \frac{1}{Z_0}\left(\frac{\delta}{i\delta j_1}\right)\left(\frac{\delta}{i\delta j_2}\right) Z_0 e^{-\frac{1}{2}j D j}\Big|_{j=0}$$

read $\frac{\delta}{i\delta j(x_1)}$ $\qquad$ read j^T

$$= -\frac{\delta}{\delta j_1}\left(-\frac{1}{2} j_x D_{x x_2} - \frac{1}{2} D_{x_2 x} j_x\right) e^{-\frac{1}{2} j D j}\Big|_{j=0}$$

$$= \frac{\delta}{\delta j_1}\left(j_x D_{x x_2}\right) e^{-\frac{1}{2} j D j}\Big|_{j=0} = D_{x_1 x_2} = D_{12} = D(x_1 - x_2)$$

Feynman rule: 

<u>Note:</u> For operators acting on fcts., we can always go to momentum space:

$$\begin{tikzcd} A: \quad f(x) \arrow[r, mapsto] \arrow[d, "\text{Fourier-trf.}"] & \int d^4y \, A(x,y) \, f(y) \arrow[d, "\text{Fourier-trf.}"] \\ \tilde{f}(p) = \int d^4x \, e^{ipx} f(x) \arrow[r, mapsto] & \int d^4q \, \tilde{A}(p,q) \, \tilde{f}(q) \end{tikzcd}$$

- From this diagram it follows that $\tilde{A}(p,q) = \int \frac{d^4x \, d^4y}{(2\pi)^4} e^{i(px-qy)} A(x,y)$.
  (Derive this!)
- Applying this to $D^{-1}(x-y) = -i(-\Box_x - m^2 + i\varepsilon)\,\delta^4(x-y)$, we get

  $\tilde{D}^{-1}(p,q) = -i(p^2 - m^2 + i\varepsilon)\,\delta^4(p-q)$ and hence

  $\tilde{D}(p,q) = \frac{i}{p^2 - m^2 + i\varepsilon}\,\delta^4(p-q)$. (Check this!)

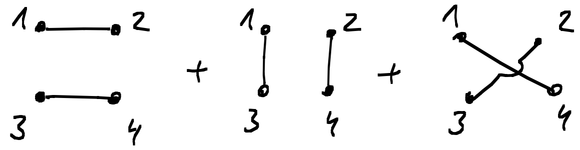
- The first, somewhat trivial, application of our Feynman rule is:

$\varphi(x_1)$ etc.

$$\langle 0|T\varphi_1\varphi_2\varphi_3\varphi_4\rangle = \frac{\delta}{i\delta j_1}\cdots\frac{\delta}{i\delta j_4}\, e^{-\frac{1}{2} jDj}\Big|_{j=0}$$

$$= D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$$

check this!

$=$ 

In words: Draw points for $x_1 \cdots x_4$ and connect them (pairwise) in all possible ways with lines. Write $D_{12} = D(x_1 - x_2)$ for a line connecting x_1 & x_2, etc.. Add all contributions (in this case 3).

2.3 $Z[j]$ for interacting theories (especially $\lambda\varphi^4$ theory)

$$Z[j] = \int D\varphi\, e^{iS_0[\varphi] + iS_{int.}[\varphi] + ij\varphi}$$

e.g. $-\frac{1}{2}\varphi D^{-1}\varphi$ e.g. $i\int d^4x\, \frac{\lambda}{4!}\varphi(x)^4$ $\int d^4x\, j(x)\varphi(x)$

Think of $e^{iS_{int}}$ as a power series in φ, i.e.

$$e^{iS_{int.}} = 1 + (-i)\int d^4x\, \frac{\lambda}{4!}\varphi(x)^4 + \frac{1}{2}(-i)\int d^4x\, \frac{\lambda}{4!}\varphi(x)^4 (-i)\int d^4y\, \frac{\lambda}{4!}\varphi(y)^4 + \cdots$$

and replace each $\varphi(x)$ by $\dfrac{\delta}{i\delta j(x)}$, acting on $e^{ij\varphi}$.

- Thus, we can formally write

$$Z(j) = \int D\varphi \, e^{iS_0[\varphi]} \, e^{iS_{int.}\left[\frac{\delta}{i\delta j}\right]} \, e^{ij\varphi}$$

$$= e^{iS_{int.}\left[\frac{\delta}{i\delta j}\right]} \, Z_0[j].$$

$$= Z_0[0] \, e^{iS_{int.}\left[\frac{\delta}{i\delta j}\right]} \, e^{-\frac{1}{2} jDj}$$

$$= Z_0[0]\left(1 - \frac{i\lambda}{4!}\int_x \left(\frac{\delta}{i\delta j(x)}\right)^4 + \cdots\right)\left(1 - \frac{1}{2} jDj + \frac{1}{2}\left(-\frac{1}{2} jDj\right)^2 + \cdots\right)$$

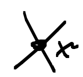
- To begin with, let's evaluate $Z[0]$, i.e. set $j=0$ <u>after</u> all $\frac{\delta}{\delta j}$ - differentiations have been performed. Ignoring the prefactor $Z_0[0]$, the term linear in λ is

$$\frac{-i\lambda}{4!}\int_x \left(\frac{\delta}{i\delta j(x)}\right)^4 \frac{1}{2}\left(-\frac{1}{2}\int_y\int_{y'} j(y) D(y-y') j(y')\right)\left(-\frac{1}{2}\int_z\int_{z'} j(z) D(z-z') j(z')\right)$$

<u>Problem:</u> Work this out explicitly!

- We can systematize the calculation by drawing pictures:

$$-\frac{i\lambda}{4!}\int_x \left(\frac{\delta}{i\delta j(x)}\right)^4 \quad \longrightarrow$$

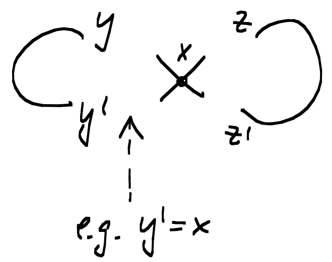


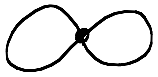
(vertex)

$$-\frac{1}{2}\int_y\int_{y'} j(y) D(y-y') j(y') \quad \longrightarrow$$

(propagator)

- Each differentiation "attaches" an end of some available "line" to the vertex associated with the differentiation!



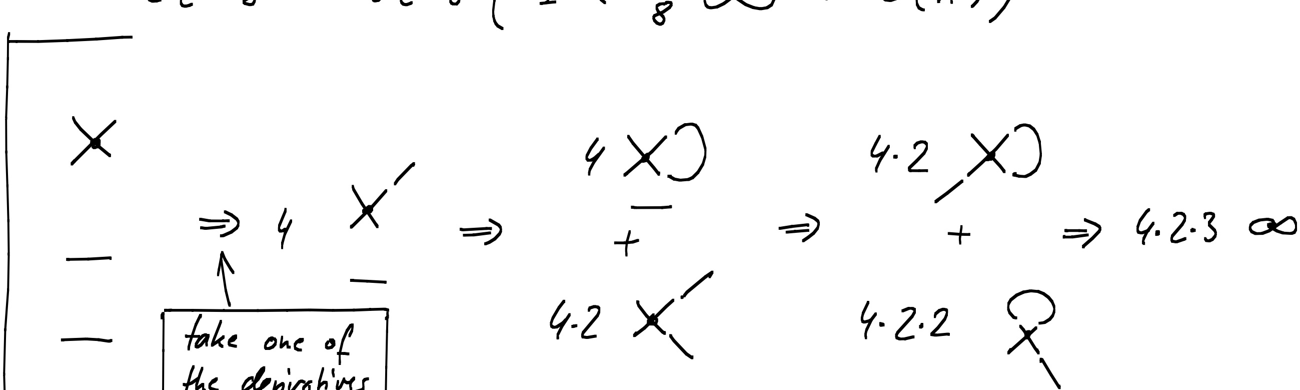
$\Rightarrow$ result:  $= -i\lambda \int d^4x \, D(x-x)\, D(x-x)$

$$= -i\lambda \int d^4x \, D(0)^2$$

(don't worry at the moment that this is totally ill-defined or divergent)

- The factor $\frac{1}{2}$ of the prop. takes care of the 2 possibilities of attaching any line to two given points.
- The factor $\frac{1}{4!}$ of the vertex takes care of the 4! possible permutations of the "attachment points" of each vertex.
- The factors $1/n!$ from the "exp" take care of permut.s of vertices & lines.
- Thus, we would expect to find a numerical prefactor 1.
- However, if a diagram has symmetries, this counting doesn't work. This (unfortunately) happens for this simplest diagram without external lines, so that we find

$$Z[0] = Z_0[0] \left(1 + \frac{1}{8} \infty + O(\lambda^2)\right)$$

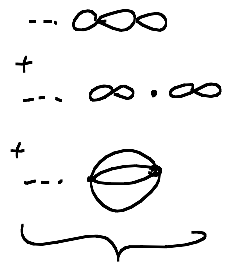


finally: $\frac{4 \cdot 2 \cdot 3}{4!\, 2!\, 2 \cdot 2} = \underline{\underline{\frac{1}{8}}}$

Thus:

$$Z[0] = Z_0[0]\left(1 + \text{sum of all "vacuum diagrams"}\right)$$
(with appropriate symm. factors)

$$= Z_0[0]\Big(1 + \underbrace{\tfrac{1}{8}\,\infty}_{\lambda} + \ldots + O(\lambda^4)\Big)$$



λ^2

The calculation of Green's fcts. is now completely straightforward:

$$\langle 0|T\varphi_1\cdots\varphi_n|0\rangle = \frac{1}{Z[0]}\left(\frac{\delta}{i\delta j_1}\right)\cdots\left(\frac{\delta}{i\delta j_n}\right)Z[j]\Big|_{j=0}$$

$$= \frac{1}{\cancel{Z_0[0]}(1+\text{vac. diagr.})}\cdot\underbrace{\left(\frac{\delta}{i\delta j_1}\right)\cdots\left(\frac{\delta}{i\delta j_n}\right)e^{iS_{int.}\left[\frac{\delta}{\delta j}\right]}}_{\substack{\text{ends of lines are either} \\ \text{external } (D(x-x_1)\text{ etc.}) \\ \text{or at a vertex}}}\cancel{Z_0[0]}\,\underbrace{e^{-\frac{1}{2}jDj}}_{\text{"lines"}}$$

- $\dfrac{1}{1+\text{vac. diagr.}}$ cancels all vac. diagrams which arise in the process of evaluating the j-differentiations.

$\Rightarrow$ <u>Feynman rules for $\lambda\varphi^4$-theory:</u>

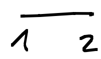
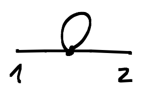
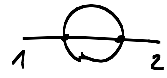
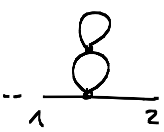
$\langle T\varphi_1\cdots\varphi_n\rangle$ = sum of all diagrams (built from $\times$ & $-$), without vac. diagrams

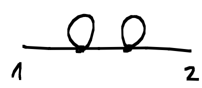
with identifications:

 $= D(x-y)$

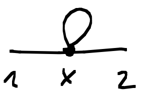
 $= -i\lambda \int d^4x$

<u>Problem:</u> Derive explicitly, using our functional methods,

$\langle T\varphi_1\varphi_2\rangle =$  $+ \cdots$  $+ \cdots$  $+ \cdots$ 

$+ \cdots$  $+ \cdots$

& determine symm. factors.

(Reminder:  $= \int d^4x\, D(x_1-x)\, D(x-x)\, D(x-x_2)$

etc.)

The generalization to other interaction-terms S_{int} and several fields is obvious. One can consider, e.g.

$$S_{int.} = \int \frac{\lambda}{n!}\varphi^n$$

or

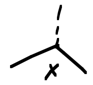
$$\left.\begin{aligned} iS_0 &= -\frac{1}{2}\varphi D_\varphi^{-1}\varphi - \frac{1}{2}\chi D_\chi^{-1}\chi \\ \text{with } S_{int} &= \int \frac{\lambda}{2}\varphi^2\chi \end{aligned}\right\}$$

need to introduce sources j^φ & j^χ for the two fields, otherwise analogous

<u>Feynman rules:</u>

 $= D_\varphi(x-y)$ for φ

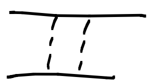
 $= D_\chi(x-y)$ for χ

 $= \int d^4x\, i\lambda$

(derive them!)

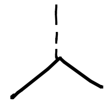
different mass

- Derive, e.g. all contributions relevant to $\varphi\varphi \to \varphi\varphi$ scattering at order λ^4:

 $+ \; ---$

- Another interesting possibility is to have derivatives in $S_{int.}$, e.g.

$$S_{int.} = \lambda \int d^4x \, \varphi^2(x) \, \partial^2 \chi(x)$$

Derive the Feynman rule for the vertex  in this case.

[in fact, this is very simple, since we always simply replace φ by $\frac{\delta}{i\delta j}$ in $S_{int.}$, i.e.

$$S_{int} \longrightarrow \frac{\lambda}{2} \int d^4x \left(\frac{\delta}{i\delta j_\varphi(x)} \right)^2 \partial^2 \left(\frac{\delta}{i\delta j_\chi(x)} \right) ;$$

we get, e.g.

$$\langle T \varphi_1 \varphi_2 \chi_3 \rangle = \int d^4x \; D_\varphi(x_1 - x) \, D_\varphi(x_2 - x) \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\mu} D_\chi(x - x_3) . \Big]$$

- A more interesting example is the complex scalar field:
  - Need 2 sources j & $\bar{J}$ for $\bar{\varphi}$ & φ

$$\Rightarrow Z_0[j, \bar{J}] = \int D\varphi \, \exp i \Big[\int d^4x \Big(\bar{\varphi} \left(-\partial^2 - m^2 + i\varepsilon \right) \varphi + \bar{\varphi} j + \bar{J} \varphi \Big] .$$

Evaluate this, introduce an interaction $\sim (\bar{\varphi}\varphi)^2$, and derive the Feynman rules (You will need to put arrows on lines, $\longrightarrow$, to keep track of j's vs. $\bar{J}$'s).