3. Fermions in the path integral approach

3.1 Bosonic harmonic oscillator in the holomorphic repres.

(also known as coherent-state or Bargmann-Fock repres.)

[cf. books by Fadeev / Slavnov;
Zinn-Justin; Itzykson / Zuber]

- Let $H_0 = \omega a^+ a$, as usual (ignore the $\frac{1}{2}\omega$-piece).

- Represent a^+ & a by $\bar{z}$ & $\frac{\partial}{\partial \bar{z}}$ acting on fcts. $f(\bar{z})$.

- Define a scalar product as

this is just $r\,dr\,d\varphi$, with $z = re^{i\varphi}$, $\bar{z} = re^{-i\varphi}$

$$\langle f_1 | f_2 \rangle = \int \frac{d^2 z}{\pi} \, \overline{f_1(\bar{z})} \, f_2(\bar{z}) \, e^{-z\bar{z}}$$

- Check that $a^+ = (a)^+$ w.r.t. this scalar product:

$$\langle f_1 | a^+ f_2 \rangle = \int \frac{d^2 z}{\pi} \, \overline{f_1(\bar{z})} \, \bar{z} f_2(\bar{z}) e^{-z\bar{z}} = \int \frac{d^2 z}{\pi} \left(\partial_z \overline{f_1(\bar{z})} \right) f_2(\bar{z}) e^{-z\bar{z}}$$

$$= \int \frac{d^2 z}{\pi} \left(\overline{\partial_{\bar{z}} f_1(\bar{z})} \right) f_2(\bar{z}) e^{-z\bar{z}} = \langle a f_1 | f_2 \rangle .$$

- An orthonormal basis is given by $\psi_n(\bar{z}) = \frac{\bar{z}^n}{\sqrt{n!}}$.

  (Check that $\langle \psi_n | \psi_m \rangle = \delta_{mn}$, using polar coordinates instead of $z, \bar{z}$.)

- Let us call $|z\rangle$ an eigenstate of a with eigenvalue z.
  (Explicitly: $|z\rangle \leftrightarrow f(\bar{y}) = e^{z\bar{y}}$).

Comment: Such states are known as "coherent states" and they are very important in considering the class. limit of a harmonic oscillator (They are as close as one gets to a "classical state".)

They are also very important in QFT since they are the QFT-realization of a situation with non-zero ("classical") field. To see this, note that

$$\text{field} \leftrightarrow a + a^{+}$$

here $\langle \bar{z}|$ is just the state dual to $|z\rangle$

$$\langle a^{+}a^{+}\ldots|\,(a+a^{+})\,|a\,a\ldots\rangle = 0 \quad \text{always.}$$

However: $\langle \bar{z}|\,(a+a^{+})\,|z\rangle = \langle \bar{z}|z\rangle(z+\bar{z}) \neq 0.$

$\Rightarrow$ These states have a non-trivial exp. value for the "field", which is not possible to realize for any Fock state with fixed particle number.

- We can explicitly identify the Hilbert space of fcts. $f(\bar{z})$ with the usual one, constructed using energy-eigenstates:

$$|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}|0\rangle \longleftrightarrow \psi_n(\bar{z}) = \frac{\bar{z}^{\,n}}{\sqrt{n!}} \longleftrightarrow |\psi_n\rangle$$

- This clearly also works for our coherent states:

$$e^{z a^{+}}|0\rangle \longleftrightarrow f_z(\bar{y}) = e^{z\bar{y}} \longleftrightarrow |z\rangle .$$

- Let us calculate the overlap of $|z\rangle$ and $|\psi_n\rangle$,

$$\langle \psi_n|z\rangle = \langle 0|\frac{a^{n}}{\sqrt{n!}}e^{z a^{+}}|0\rangle = \frac{z^{n}}{\sqrt{n!}},$$

and between two coherent states:

$$\langle \bar{z}|y\rangle = \sum_n \langle \bar{z}|\psi_n\rangle\langle\psi_n|y\rangle = \sum_n \frac{(\bar{z}y)^{n}}{n!} = e^{\bar{z}y}$$

(This "$-$" is just notation, $\langle \bar{z}|$ is the dual state belonging to $|z\rangle$)

- Crucial claim: $\mathbb{1} = \int \frac{d^2z}{\pi}\,|z\rangle\langle\bar{z}|\,e^{-\bar{z}z}$

Demonstration:

$$\int \frac{d^2z}{\pi} \langle \psi_n | z \rangle \langle \bar{z} | \psi_m \rangle e^{-z\bar{z}} \underset{\uparrow}{=} \delta_{mn} = \langle \psi_n | \mathbb{1} | \psi_m \rangle$$

cf. the calculation leading to $\langle \psi_n | \psi_m \rangle = \delta_{nm}$ $\square$.

Note: The states $|z\rangle$ are overcomplete (cf. $\langle \bar{y} | z \rangle = e^{\bar{y} z}$), but the representation of $\mathbb{1}$ using an integral over $|z\rangle\langle\bar{z}|$ nevertheless exists (and will be crucial below).

3.2 Path integral in coherent (holomorphic) states

$$\langle \bar{z} | e^{-iHt} | y \rangle = \prod_{i=1}^{n-1} \left(\int \frac{d^2 z_i}{\pi} e^{-z_i \bar{z}_i} \right) \cdot \langle \bar{z} | e^{-iH\varepsilon} | z_{n-1} \rangle \langle \bar{z}_{n-1} | e^{-iH\varepsilon} | z_{n-2} \rangle$$

$$\ldots\ldots \langle \bar{z}_1 | e^{-iH\varepsilon} | y \rangle ,$$

with $\varepsilon = \frac{t}{n}$.

- Let $H = H(a^+, a)$ be given as a (normal ordered) fct. of a^+ & a.

  Then $\langle \bar{z}_i | e^{-iH(a^+,a)\varepsilon} | z_{i-1} \rangle$

  $$\simeq \langle \bar{z}_i | (1 - iH(a^+,a)\varepsilon) | z_{i-1} \rangle = e^{\bar{z}_i z_{i-1}} (1 - iH(\bar{z}_i, z_{i-1}) \cdot \varepsilon)$$

  $$\simeq e^{\bar{z}_i z_{i-1} - iH(\bar{z}_i, z_{i-1})\varepsilon} .$$

- Calling $\bar{z} = \bar{z}_n$ & $y = z_0$, we finally get

$$\langle \bar{z}_n | e^{-iHt} | z_0 \rangle = \prod_{i=1}^{n-1} \left(\int \frac{d^2 z_i}{\pi} \right) \cdot \exp\Big[\bar{z}_n z_{n-1} - \bar{z}_{n-1} z_{n-1} + \bar{z}_{n-1} z_{n-2} - \cdots$$

$$+ \bar{z}_1 z_0 - i\varepsilon \sum_{i=1}^{n} H(\bar{z}_i, z_{i-1}) \Big]$$

- The explicit sum of "$\bar z z$-terms" can be written as

$$\varepsilon \sum_{i=2}^{n} \frac{\bar z_i - \bar z_{i-1}}{\varepsilon} z_{i-1} + \bar z_1 z_0 \quad ,$$

which allows us to take (formally) the continuum limit:

$$\langle \bar z_b | e^{-iHt} | z_a \rangle = \underset{\uparrow}{\int} Dz\, D\bar z \, \exp\Big[\bar z(0) z(0) + \int_0^t dt' \Big(\dot{\bar z} z - iH(\underset{\uparrow}{\bar z}, \underset{\uparrow}{z})\Big)\Big]$$

for a^+, a

we view this (formally)* as an integral over <u>independent</u> functions $z(t')$, $\bar z(t')$, with $z(0) = z_a$ & $\bar z(t) = \bar z_b$

(In principle, this requires a careful reconsideration of the previous derivation, with more emphasis on analytic fcts., Laplace-transform etc. ...)

- The exponent can be given in a more symmetric form using integration by parts:

$$\exp\Big[\underbrace{\tfrac{1}{2}\big(\bar z(t) z(t) + \bar z(0) z(0)\big)}_{} + \underbrace{\int_0^t dt' \Big\{\tfrac{1}{2}\big(\dot{\bar z} z - \bar z \dot z\big) - iH(\bar z, z)\Big\}}_{}\Big]$$

These are extra "boundary terms" arising because of the fact that we fix z at $t'=0$ & $\bar z$ at $t'=t$.

Up to further boundary terms, this is just iS, as in the previously derived path-integral formulation

<u>Demonstration:</u>

$$q = \frac{1}{\sqrt{2\omega}}(a + a^+) \qquad\qquad a = \tfrac{1}{2}\Big(\sqrt{2\omega}\, q + i\sqrt{\tfrac{2}{\omega}}\, p\Big) = z$$

$$\Longrightarrow$$

$$p = -i\sqrt{\tfrac{\omega}{2}}(a - a^+) \qquad\qquad a^+ = \tfrac{1}{2}\Big(\sqrt{2\omega}\, q - i\sqrt{\tfrac{2}{\omega}}\, p\Big) = \bar z$$

$$\dot{\bar z} z = \frac{\omega}{2} q \dot q - \frac{i}{2} \dot p q + \frac{i}{2} p \dot q + \frac{1}{2\omega} p \dot p$$

$$\frac{1}{2i}\left(\dot{\bar z} z - \bar z \dot z\right) = -\frac{1}{2} \dot p q + \frac{1}{2} p \dot q = -\frac{1}{2} \frac{d}{dt}(pq) + p \dot q$$

$$\Rightarrow \int_0^t dt' \left\{ \frac{1}{2} \left(\dot{\bar z} z - \bar z \dot z\right) - i H(z,\bar z) \right\} = i \int_0^t dt' \left(p \dot q - H(p,q) \right) + \text{Boundary terms}$$

$$= i \underbrace{S[p,q]}$$

This is the action of the Hamilton-formulation of classical mechanics, which gives the Hamilton-eqs. if we vary $q(t)$ and $p(t)$ as independent fcts.

<u>Final result:</u>

$$\langle 0 | T a(t_1)\, a^+(t_2) \cdots | 0 \rangle = \frac{\int Dz D\bar z \left[z(t_1)\, \bar z(t_2) \cdots \right] \exp i S[z,\bar z]}{\int Dz D\bar z \exp i S[z,\bar z]}$$

$\uparrow$ $\uparrow$ Heisenberg-picture operators ; $\langle 0 | 0 \rangle = 1$ assumed

$\uparrow$ with an appropriate convergence factor $i\varepsilon$ included, as before.

$S[z,\bar z]$ is obtained by deriving S from some Hamilton $H(a, a^+)$ and replacing a, a^+ by the class. variables $z, \bar z$.

(We will not restrict ourselves to the case $H = \omega a a^+$, but allow extra terms with higher powers of a & a^+, which we however automatically consider to be normal-ordered.)

The "extra boundary terms" will never play a role since we will always integrate from $t = -\infty$ to $t = +\infty$.

*) This formal treatment of $z, \bar{z}$ as indep. variables can be justified better:

$$\int d^2z \sim \int dz\, d\bar{z} \sim \int dx\, dy = \underbrace{\int_{-\infty}^{+\infty} dx}\, \underbrace{\int_{-\infty}^{+\infty} dy}$$

integrals over the contours $\operatorname{Im} x = 0$ & $\operatorname{Im} y = 0$ for two indep. complex variables x & y.

Using $z = x + iy$ / $\bar{z} = x - iy$, we can now also think of $\int dz\, d\bar{z}$ as integrals over two contours C_1 & C_2 for indep. variables $z, \bar{z}$ under the condition $\bar{C}_1 = C_2$. Furthermore, we can deform C_2 to be $C_2 \neq \bar{C}_1$, so that we are now really performing two 1-dim. integrals over indep. complex variables $z, \bar{z}$.

(This can be of practical relevance even in the simple q.m. case if $S(z, \bar{z})$ has a stationary point for a path which does <u>not</u> fulfil $\bar{z} = (z)^*$.)

3.3 Fermionic harmonic oscillator & Grassman variables

- Consider $H_0 = \omega a^+ a$ with $\{a, a^+\} = 1$
  $\{a, a\} = 0$ (i.e. $a^2 = a^{+2} = 0$)

- We would like to repeat our previous derivation of a path-integral formula, integrating over "class." variables $\theta, \bar{\theta}$ replacing a, a^+.

- To achieve this we think of analytic fcts. in $z, \bar{z}$ as formal power series in z & $\bar{z}$. We generalize this to "analytic fcts of Grassman variables" θ & $\bar{\theta}$, where we demand the algebraic (anticommut.) relations $\{\theta,\bar\theta\} = 0 \quad (= \theta\bar\theta + \bar\theta\theta)$

$\{\theta,\theta\} = 0 \quad$ (i.e.

$\{\bar\theta,\bar\theta\} = 0 \quad \theta^2 = \bar\theta^2 = 0).$

- We realize a^+ & a as $\bar\theta$ and $\frac{\partial}{\partial\bar\theta}$, acting on analytic fcts. of $\bar\theta$ only. Clearly, the Hilbert space is just 2-dim. since $\bar\theta^2 = 0$:

$$f(\bar\theta) = f_0 + f_1\bar\theta .$$

- With the natural definition $\frac{\partial}{\partial\bar\theta}(f_0 + f_1\bar\theta) = f_1$, we find:

$$\left(\frac{\partial}{\partial\bar\theta}\right)^2 = 0 \; ; \; \bar\theta^2 = 0 \; ;$$

$$\left\{\frac{\partial}{\partial\bar\theta}, \bar\theta\right\} f(\bar\theta) = \frac{\partial}{\partial\bar\theta}\bar\theta f(\bar\theta) + \bar\theta\frac{\partial}{\partial\bar\theta} f(\bar\theta)$$

$$= f_0 + \bar\theta f_1 \; , \text{ i.e. } \quad \left\{\frac{\partial}{\partial\bar\theta}, \bar\theta\right\} = 1 \; , \text{ as required.}$$

- Thinking of the harmonic oscill. interpretation, it is natural to write

$$|f\rangle = |0\rangle f_0 + |1\rangle f_1$$

and define the scalar product as

$$\langle g|f\rangle \equiv \bar g_0 f_0 + \bar g_1 f_1 .$$

- To realize this as an integral, we need to define Grassmann variable integration (Berezin integral):

$$\int d\theta \cdot 1 = \int d\bar\theta \cdot 1 = 0 \; ; \; \int d\theta \cdot \theta = 1 \; ; \; \int d\bar\theta\, \bar\theta = 1 .$$

(If one wants to ensure $\int d\theta \frac{\partial}{\partial\theta}(\cdots) = 0$, which is indeed a very important relation, one basically has no other choice (up to normalization).)

<u>Note:</u> $\int d\theta\,(\cdots) = \frac{\partial}{\partial\theta}(\cdots)$ etc. & $\int d\theta\, d\bar\theta = -\int d\bar\theta\, d\theta .$

- It is easy to check that

$$\langle g | f \rangle \equiv \int d\bar{\theta} d\theta \, \overline{g(\bar{\theta})} \, f(\bar{\theta}) \, e^{-\bar{\theta}\theta}$$

is consistent with the conventional definition given above and that $a^+ = (a)^+$. (Check this!)

- As before, we have $|\eta\rangle \leftrightarrow f(\bar{\xi}) = e^{\bar{\xi}\eta} = 1 + \bar{\xi}\eta$

$$= |0\rangle + |1\rangle\eta$$

(Here we have allowed for lin. combinations of Hilbert space vectors with Grassmann-valued coefficients. More mathematically: We work in the tensor product of the Hilbert space and the Grassmann algebra.)

- Crucial fact: $a|\eta\rangle = a(|0\rangle + |1\rangle\eta) = |0\rangle\eta \underset{\substack{\uparrow \\ \text{This holds since } \eta^2 = 0}}{=} (|0\rangle + |1\rangle\eta)\eta = \eta|\eta\rangle$

- In deriving the path integral formula, the crucial relation in the bosonic case was $\mathbb{1} = \int \frac{d^2z}{\pi} |z\rangle\langle\bar{z}| \, e^{-\bar{z}z}$.

- Now, we have analogously

$$\mathbb{1} = \int d\bar{\theta} d\theta \, e^{-\bar{\theta}\theta} \, |\theta\rangle\langle\bar{\theta}| .$$

Demonstration: Note : $\langle f | \theta \rangle = \bar{f}_0 + \theta \bar{f}_1$

$$\langle \bar{\theta} | g \rangle = g_0 + \bar{\theta} g_1$$

$$\langle f | g \rangle = \langle f | \mathbb{1} | g \rangle = \int d\bar{\theta} d\theta \, e^{-\bar{\theta}\theta} (\bar{f}_0 + \theta \bar{f}_1)(g_0 + \bar{\theta} g_1)$$

$$= \bar{f}_0 g_0 + \bar{f}_1 g_1 \qquad \square .$$

- With these tools in our hands, we can now repeat the derivation of the path int. formula for the bosonic harmonic oscillator step by step. (Do that !) [Order of θ & $\bar{\theta}$ in terms like $e^{-\bar{\theta}\theta}$ is important !]

The result is

$$\langle 0 | T a(t_1) a^+(t_2) \cdots | 0 \rangle = \frac{\int D\bar{\theta} D\theta \left(\theta(t_1) \bar{\theta}(t_2) \cdots \right) \exp i S[\theta, \bar{\theta}]}{\int D\theta D\bar{\theta} \, \exp \underset{\substack{\uparrow \\ \text{with } i\varepsilon\text{ - correction}}}{i} S[\theta, \bar{\theta}]}$$

<u>Note:</u> Since we manipulated $\int d\bar{\theta}(t_i) d\theta(t_i)$ etc. in purely formal, algebraic way, it is trivial to assume that θ & $\bar{\theta}$ are independent variables.

[In many cases, like in our harmonic-oscillator example, the space of analytic fcts. in $\theta, \bar{\theta}$ carries a natural $*$- operation defined by $(\theta)^* = \bar{\theta}$, $(\theta\bar{\theta})^* = (\bar{\theta})^* \theta^* = \theta\bar{\theta}$ etc. However, such a star-operation represents an extra structure, which does not have to be present in general.]

3.4 Path integral for fermions in QFT

- Recall our discussion of free Dirac fermions:

$$\mathcal{L} = \bar{\psi} (i \not{\partial} - m) \psi \ ; \qquad \psi = \{ \psi_a \} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_4 \end{pmatrix}$$

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = i \psi_a^+ \quad \text{or} \quad \pi = i \psi^+ \text{ (interpreting } \pi \text{ as a row-vector).}$$

$$H = \int d^3\bar{x} \, \mathcal{H} = \int d^3\bar{x} \left(\pi \dot{\psi} - \mathcal{L} \right) = \int d^3\bar{x} \left(i \psi^+ \dot{\psi} - \mathcal{L} \right)$$

$$= \int d^3\bar{x} \, \psi^+ (-i\gamma^i \partial_i + m) \psi = \int d^3\bar{x} \, d^3\bar{y} \sum_{a,b=1}^{4} \psi_a^+(\bar{x}) \underbrace{\left[-i(\gamma^0\gamma^i)_{ab} \frac{\partial}{\partial x^i} + m\gamma^0_{ab} \right] \delta^3(\bar{x}-\bar{y})}_{\substack{\text{We view this as a matrix with} \\ \text{"indices" } \bar{x}, a \ \& \ \bar{y}, b.}} \psi_b(\bar{y})$$

- We quantized it with the postulated (anti-) commut. relations

$$\{\psi(\bar{x}), \pi(\bar{y})\} = i\delta^3(\bar{x}-\bar{y})\cdot \mathbb{1} \quad , \text{ i.e. } \quad \{\psi_a(\bar{x}), \psi_b^+(\bar{y})\} = \delta^3(\bar{x}-\bar{y})\,\delta_{ab} .$$

(and all $\psi - \psi$ & $\psi^+ - \psi^+$ anticommut. $= 0$).

- Hence, we are precisely in the framework of a set of the fermionic harmonic oscillator discussed above.

$$H = \omega a^+ a \quad \longrightarrow \quad H = \sum_{ij} a_i^+ M_{ij} a_i \quad ,$$

with the only difference that $i \longrightarrow \{\bar{x}, a = 1 \cdots 4\}$ and that our variables are now called ψ^+ & ψ.
(They are not the creation/annih. operators of the QFT!)

- Our derivation of the path integral formula goes through without change, just with writing sums over indices at each step. The crucial point is that the anticommut. relations are those of an indep. set of oscillators,

$$\{a_i^+, a_j\} = \delta_{ij} \quad \rightarrow \quad \{\psi^+(\bar{x}), \psi(\bar{y})\} = \delta^3(\bar{x}-\bar{y})\,\mathbb{1} .$$

The fact that $H = a_i^+ M_{ij} a_j \neq a_i^+ \delta_{ij} a_j$ does clearly not affect the calculation at any point. (We really never needed the explicit form of H – it just has to be "normal ordered" in ψ^+ & ψ.)

Thus: $$\langle T \psi_{a_1}(x_1) \cdots \bar{\psi}_{b_1}(y_1) \cdots \rangle = \frac{\int D\bar{\psi} D\psi \; \psi_{a_1}(x_1) \cdots \psi_{b_1}(y_1) \cdots e^{iS}}{\int D\bar{\psi} D\psi \; e^{iS}}$$

$$S = \int d^4x \, \bar{\psi}(i\not{\partial} - m + i\varepsilon)\psi$$

Note: We write $D\bar{\psi}$ rather than $D\psi^+$ – this is only a trivial change of variables since $\bar{\psi} = \psi^+\gamma^0$.

Important conceptual comment:

- In the bosonic case, we have started from $H = :H(a,a^+):$ and derived a path-int. formula for time-ordered vacuum corr. fcts. based on the action $$S = \int dt \left(i\bar{z}\dot{z} - H(z,\bar{z}) \right)$$ (ignoring bound. terms).

- Analogously, in the fermionic case, we started from $H = :H(a,a^+):$ (where now $\{a,a^+\} = 1$ instead of $[a,a^+] = 1$) and derived a path-int. formula for time-ordered vacuum corr. fcts. based on the action $$S = \int dt \left(i\bar{\theta}\dot{\theta} - H(\theta,\bar{\theta}) \right)$$ (again ingnoring bound. terms).

  Now, however, our "classical" variables $\theta, \bar{\theta}$ are Grassmann variables and integrations are done using the formal Berezin integral that was defined above.

- QFT with Dirac spinors is a straightforward multi-variable extension of the above, with $\theta, \bar{\theta} \to \psi_a(\bar{x}), \psi^+_a(\bar{x})$.

  Hence, we have demonstrated the equivalence of the our canonical quantization of Dirac spinors in QFT I with the path-integral formulation given above, where our action is $$S = \int dt \left[\int d^3\bar{x} \sum_{a=1}^{4} \psi^+_a \dot{\psi}_a - H[\psi,\psi^+] \right].$$

  (Here $\psi_a(\bar{x}), \psi^+_a(\bar{x})$ are independent Grassmann variables for every $\bar{x}$ & a. We have simply not performed the name change $a \to \theta$.)

  This action happens to be precisely $S = \int d^4x\, \bar{\psi}(i\not{\partial} - m)\psi$, as we have already seen earlier.

- One subtlety which we so far ignored is that we should actually use an operator H normal-ordered in ψ/ψ^+ rather than in $a_{\bar{k}}, b_{\bar{k}} / \bar{a}_{\bar{k}}^+, b_{\bar{k}}^+$ in the argument given above. However, this is indeed irrelevant since it only leads to an additive constant, which has no physical significance.

The $i\varepsilon$-term

In the fermionic case, the $i\varepsilon$-term can not simply be viewed as a convergence factor, since we are dealing with a formally-defined integral (i.e. $\int d\bar{\theta} a\theta \, e^{a\bar{\theta}\theta}$ is well-defined for any a). We need to check that this term suppresses non-vacuum states exponentially, so that we really get vacuum corr. fcts. We propose that the correct term arises from $m \to m - i\varepsilon$. Since the Hamiltonian contains $m\bar{\psi}\psi$, we get

$$-i\varepsilon \int d^3\bar{x}\, \bar{\psi}\psi \;=\; -i\varepsilon \int d\tilde{p}\, \frac{m}{p_0}\left(a_{\bar{p}}^{s+} a_{\bar{p}} - b_{\bar{p}}^{s} b_{\bar{p}}^{s+}\right)$$

$$\left[\text{recall that } \psi(\bar{x}) = \int d\tilde{p}\left(a_{\bar{p}}^{s} u_s(p) e^{i\bar{p}\bar{x}} + b_{\bar{p}}^{s} v_s(p) e^{-i\bar{p}\bar{x}}\right)\right.$$

$$\left. \& \;\; \bar{u}_r(p) u_s(p) = 2m\delta_{rs} \;\; \text{etc.}\right]$$

$$\ldots = \underbrace{-i\varepsilon \int d\tilde{p}\, \frac{m}{p_0}\left(a_{\bar{p}}^{s+} a_{\bar{p}} + b_{\bar{p}}^{s+} b_{\bar{p}}^{s}\right)}_{\text{positive on non-vacuum states}} + \text{irrelevant constant}$$

$$\Rightarrow\; e^{-iH\Delta t} \supset e^{-i(-i\varepsilon)\Delta t\cdot(\text{positive})} = e^{-\Delta t\cdot(\text{positive})}$$

$\Rightarrow$ our proposal $m \to m - i\varepsilon$ is indeed the correct choice, in agreement with what was found in the canonical approach in QFT I. □

- As in the bosonic case, we can introduce 'source fields' $\eta(x)$, $\bar\eta(x)$ (which are spinors & Grassmann variables, just like $\psi(x)$, $\bar\psi(x)$) and define:

$$Z[\bar\eta,\eta] \equiv \int D\bar\psi D\psi\, e^{iS + \underset{\uparrow}{i\bar\eta\psi} + i\bar\psi\eta}$$

read: $i\int d^4x\,\bar\eta(x)\psi(x)$.

- We find:

$$\langle T\psi_1\bar\psi_2 \cdots\rangle = \frac{1}{Z}\left(\frac{\delta}{i\delta\bar\eta_1}\right)\underset{\uparrow}{\left(\frac{\delta}{-i\delta\eta_2}\right)}\cdots\, Z[\bar\eta,\eta]\Big|_{\bar\eta,\eta=0}$$

<u>Note:</u> We introduce an extra minus-sign to ensure that

$$\frac{\delta}{-i\delta\underset{\nwarrow}{\eta_2^b}}(i\bar\psi\eta) = \int d^4x\;\bar\psi^a(x)\underbrace{\frac{\delta}{\delta\eta^b(x_2)}\eta^a(x)}_{\delta^{ab}\delta^4(x_2-x)} = \underset{\substack{\uparrow\\ !}}{+}\bar\psi^b(x_2)$$

spinor index

- As before, the free case is easy to treat completely explicitly:

$$S_0 = \bar\psi(i\not\partial - m)\psi$$ (we suppress $i\varepsilon$ when we don't need it)

$$iS_0 = -\bar\psi S^{-1}\psi$$

where now $S^{-1}(x,y) = \frac{1}{i}(i\not\partial_x - m)\,\delta^4(x-y)$

& $S(x-y) = \int d^4k\,\frac{i}{\not k - m}\,e^{-ik\cdot(x-y)}$.

$$Z_0[\bar\eta,\eta] = \int D\bar\psi D\psi\,\exp[-\bar\psi S^{-1}\psi + i\bar\eta\psi + i\bar\psi\eta]$$

<u>substitution:</u> $\psi \to \psi + iS\eta$; $\bar\psi \to \bar\psi + i\bar\eta S$

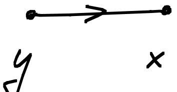
$$\Rightarrow Z_0[\bar\eta,\eta] = \int D\bar\psi D\psi \exp[-\bar\psi D^{-1}\psi] \exp[-\bar\eta S\eta]$$

$$= \underline{\underline{Z_0[0,0]\, e^{-\bar\eta S\eta}}}.$$

- This gives rise to the Feynman rule for the propagator as in the bosonic case:

$$\langle T\psi(x)\bar\psi(y)\rangle = \frac{\delta}{i\delta\bar\eta(x)}\cdot\frac{\delta}{-i\delta\eta(y)}\, e^{-\bar\eta S\eta} = S(x-y)$$

diagram:



Comment: The shift of the integration variable is easy to justify in the Grassmann case:

$f(\theta)=f_0+f_1\theta$; $\int d\theta\, f(\theta) = f_1$

$$\int d\theta\, f(\theta-\eta) = \int d\theta\,(f_0 + f_1\theta - f_1\eta) = f_1 \quad ✓$$

new "f_0"

Comment:

It is straightforward to introduce interactions, e.g. with a bosonic field

$$iS = -\bar\psi S_F^{-1}\psi - \frac{1}{2}\varphi D_F^{-1}\varphi \overbrace{-i\lambda\bar\psi\psi\varphi}^{iS_{int}}$$

and to derive Feynman rules using the trick of sources &

$$S_{int.}[\bar\psi,\psi,\varphi] \longrightarrow S_{int.}\left[\frac{\delta}{-i\delta\eta},\frac{\delta}{i\delta\bar\eta},\frac{\delta}{i\delta j}\right].$$

(Work this out for the above "Yukawa-type" interaction!)

3.5 The "fermionic determinant"

- Before closing this chapter, let us calculate some simple bosonic & fermionic (path) integrals explicitly.

- We start with the trivial bosonic example

$$\int d^2z \, e^{-a z\bar{z}} \sim \int r dr d\varphi \, e^{-ar^2} \sim \frac{1}{a} \, .$$

- The analogous fermionic integral gives

$$\int d\bar{\theta} d\theta \, e^{-a\bar{\theta}\theta} = \int d\bar{\theta} d\theta \, (1 - a\bar{\theta}\theta) = a \, .$$

- This very important "inversion" continues to hold for many variables. We again start with the bosonic case:

$$\prod_i \left(\int d^2 z_i \right) e^{-\bar{z}_j A_{jk} z_k}$$ (A - hermitian matrix, which ensures reality of exponent)

$$\sim \prod_i \int (d^2 z_i') \, |\det U|^2 \exp(-\bar{z}_j' U^+_{jk} A_{kl} U_{lm} z_m') \, ,$$

where we used $z_i = U_{ik} z_k'$; U - unitary

$$\prod_i \left(\int dz_i \right) = \prod_i \left(\int dz_i' \right) \det U$$ & analogously for $\bar{z}$.

Since $|\det U| = 1$ and since U can be chosen to make U^+AU diagonal, we immediately see

$$\prod_i \left(\int d\bar{z}_i dz_i \right) e^{-\bar{z}_i A_{jk} z_k} \sim \frac{1}{\det A} \, .$$

- We now repeat this for Grassmann variables ("fermions"):

$$\prod_i \left(\int d\bar{\theta}_i d\theta_i \right) e^{-\bar{\theta}_j A_{jk} \theta_k} = \prod_i \left(\int d\bar{\theta}_i d\theta_i \right) \frac{1}{n!} (-\bar{\theta}_{j_1} A_{j_1 k_1} \theta_{k_1}) \cdots (-\bar{\theta}_{j_n} A_{j_n k_n} \theta_{k_n})$$

$$= \prod_i \left(\int d\bar\theta_i \, d\theta_i \right) \frac{1}{n!} \Big\{ \underbrace{(-\bar\theta_1 A_{11} \theta_1) \cdots (-\bar\theta_n A_{nn} \theta_n) + \cdots + \cdots\cdots\cdots}_{} \Big\}$$

$(n!)^2$ terms resulting from the first one by allowing for all possible permutations of the first indices as well as for all possible permutations of the second indices

$$= \frac{1}{n!} \Big\{ A_{11} A_{22} \cdots A_{nn} + \cdots \Big\} \qquad \left((n!)^2 \text{ terms, as before} \right)$$

$$= \underline{\underline{\det A}} \qquad \Big(\text{cf. the usual definition}$$

$$\varepsilon_{i_1 \cdots i_n} A_{i_1 j_1} \cdots A_{i_n j_n} \varepsilon_{j_1 \cdots j_n} = n! \det A \Big)$$

$\uparrow$

This is precisely the inverse of the bosonic result.

Note: This immediately applies to QFT, where we have, for a complex scalar:

$$Z_0[\bar j, j]\Big|_{j = \bar j = 0} = \int D\bar\varphi \, D\varphi \; e^{-\bar\varphi D^{-1} \varphi} \sim \frac{1}{\det D^{-1}}$$

& for a Dirac fermion:

$$Z_0[\bar\eta, \eta]\Big|_{\bar\eta = \eta = 0} = \int D\bar\psi \, D\psi \; e^{-\bar\psi S^{-1} \psi} \sim \det S^{-1} .$$

Note: The expressions on the r.h. side are not as meaningless as they appear:

$$\text{e.g. (bosonic)} \quad \ln Z_0[0] = -\ln \det D^{-1} = -\operatorname{tr} \ln D^{-1}$$

$$\sim -\underbrace{\int d^4k}_{\text{"tr"}} \ln(k^2 + m^2) \quad ;$$

This can be regularized and, in many cases, meaningfully calculated.