6 QCD – Running Coupling & β-Function

6.1 General Structure & Running Coupling

- $SU(3)$ gauge-th. with (several generations) of fermions (quarks) in fund. repres.:

$$\mathcal{L} = -\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \sum_f \bar{\psi}_f (i\not{D} - m) \psi_f$$

$$f \in \{\underbrace{u, d, s}_{\text{"light"}}, \underbrace{c, b, t}_{\text{"heavy"}}\}$$

- $A_\mu = A_\mu^a T^a \; ; \quad T^a = \frac{1}{2}\lambda^a$ ← "Gell-Mann matrices"

$$\lambda^a = \left(\begin{array}{c|c} \sigma^a & \\ \hline & \end{array}\right) \; ; \quad a = 1..3$$

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} ; \; \lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix} ; \; \lambda^6 = \left(\begin{array}{c|c} \; & \\ \hline & \sigma^1 \end{array}\right) ; \; \lambda^7 = \left(\begin{array}{c|c} \; & \\ \hline & \sigma^2 \end{array}\right)$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \qquad \left([\lambda^3, \lambda^8] = 0 \; ; \; \text{"Cartan subalgebra"} \right)$$

- Let us do pert. theory (according to Feynm. rules of Ch. 5), using <u>dimensional regularization</u> (cf. also Ch. 11 of QFT I)

- In d dims., g^2 is not dim. less:

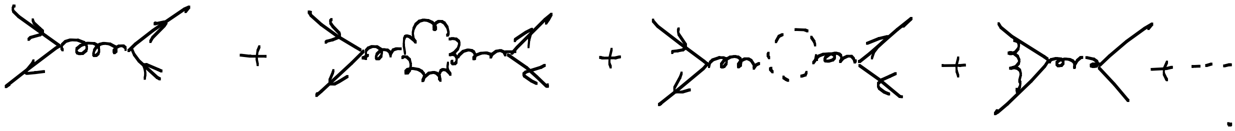
$$D_\mu = \partial_\mu + i A_\mu \quad \Rightarrow \quad [A_\mu] = 1$$

$$S = \int d^dx \, \frac{1}{2g^2} \operatorname{tr}(F^2) \quad \Rightarrow \quad -d - [g^2] + 2 + 2[A_\mu] = 0$$

$$\Rightarrow \quad [g^2] = 4 - d$$

- We write $d = 4 - \epsilon$ and redefine $g^2 \to g^2 \mu^\epsilon$, where μ with $[\mu] = 1$ is a new parameter.

- Calculate some dim.less phys. observable (e.g. σ for $q\bar{q} \to q\bar{q}$, for heavy quarks of mass m at c.m.s. energy Q ($s = Q^2$), normalized by approp. power of Q)



- Result:

$$R = R(Q^2/\mu^2, g^2)$$

$$= \left[g^2 \mu^\epsilon + g^4 \mu^{2\epsilon} \underbrace{f(Q^2, \epsilon)} + O(g^6) \right] Q^{-\epsilon}$$

Prefactor 1 for simplicity

This follows from Feynman diagrams (1-loop) in $4-\epsilon$ dims. ; μ does not appear!

$$\Rightarrow f(Q^2, \epsilon) = Q^{-\epsilon} \left(c_1 \frac{1}{\epsilon} + c_2 + c_3 \epsilon + \dots \right)$$

[A typical loop-integral is

$$\int \frac{d^d k}{(k^2 + Q^2)^2} \sim Q^{d-4} \int \frac{d^d x}{(x^2+1)^2} \sim Q^{-\epsilon} \left(\dots \frac{1}{\epsilon} + \dots \right)$$]

$$\Rightarrow R = \left[g^2 + g^4 e^{\epsilon \ln \mu/Q} \left(c_1 \frac{1}{\epsilon} + c_2 + O(\epsilon) \right) \right] \left(\frac{\mu}{Q} \right)^\epsilon$$

$$= \left[g^2 + g^4 \left(c_1 \frac{1}{\epsilon} + c_2 + c_1 \ln \mu/Q + O(\epsilon) \right) \right] \left(\frac{\mu}{Q} \right)^\epsilon$$

- We now need to renormalize. A very convenient renormalization scheme is "minimal subtraction":

- Redefine $g = g_{bare}$ as $g_{bare} = g_{phys.} Z_g$.

  Z_g must be a fct. of ϵ, chosen such that R remains finite as $\epsilon \to 0$. We chose:

$$\boxed{Z_g = 1 + \frac{\#}{\epsilon}}$$

("minimal", since just the pole is subtracted.)

- In our case, obviously $Z_g = 1 - \frac{1}{2} c_1 \frac{1}{\epsilon}$.

- We find: (at $\epsilon \to 0$ and with $g_{phys.} \to g$)

$$R = \left[g^2 \left(1 + g^2 (c_2 + c_1 \ln \mu/Q)\right) + O(g^4) \right] \underbrace{\left(\frac{\mu}{Q}\right)^\epsilon}_{=1 \text{ since } \epsilon \to 0}$$

(Beware, from now on g is our phys. coupling.)

- This works to all orders in g and we find

$$\underline{\underline{R = R(g^2, Q^2/\mu^2)}}$$

[Note: We have chosen not to renormalize our fields, i.e. not to substitute $A_\mu \to A_\mu Z_A^{1/2}$. In this way, we don't get finite Greens fcts., but phys. observables can nevertheless all be made finite. A redefinition of fields would not affect the result since it affects both the vertices and the Z-factors of LSZ. These effects cancel in cross sects.]

- Obviously, the MS-scheme is, in fact, a 1-parameter family of schemes (with the parameter being μ).

- Of course, R can not depend on which scheme one uses, i.e., it can not depend on μ. This can only be the case if $g = g_{phys.}$ depends on μ, i.e.

$$0 = \underbrace{\mu^2 \frac{d}{d\mu^2}}_{\frac{d}{d\ln\mu^2}} R\left(g^2(\mu^2),\ Q^2/\mu^2\right)$$

$$\Rightarrow\ 0 = \mu^2\left(\frac{\partial}{\partial\mu^2} + \frac{dg^2}{d\mu^2}\frac{\partial}{\partial g^2}\right) R\left(g^2, Q^2/\mu^2\right)$$

or $$\mu^2\frac{dg^2}{d\mu^2} = \underbrace{-\left(\mu^2\frac{\partial}{\partial\mu^2}R\right)\Big/\left(\frac{\partial}{\partial g^2}R\right)}$$

This expression does not depend on Q^2/μ^2 (since it describes the change of the coupling with the change of the renorm. scheme — — this should not depend on the observable we were calculating.)

$$\Rightarrow\ \mu\frac{dg}{d\mu} = \underbrace{\beta(g)}$$

calculable order-by-order in g.

- In our example:

$$\frac{dg^2}{d\ln\mu^2} = -\frac{g^4\cdot\frac{c_1}{2}}{1} + O(g^6)$$

we can only use the leading order since numerator is leading order.

$$\Rightarrow \beta(g) = -\frac{c_1}{2} g^3 + \cdots .$$

Fact: (to be derived later)

For $SU(N_c)$ gauge-th. with N_f flavors:

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$$

$$\left(\text{cf. } \beta(e) = \frac{e^3}{12\pi^2}\right)$$

Problem: Can you get $\beta(e)$ if you are given $\beta(g)$, without doing any loop calc.?

- To get an optimal result at fixed order, chose $\mu = Q$ (to avoid "large logs").

$$\Rightarrow R = R(g^2(Q^2), 1)$$

- Actually, the true expansion parameter is

$$\frac{\alpha_s}{4\pi}\cdot N_c = \underbrace{\frac{g^2}{16\pi^2}}_{\text{"loop factor"}} \underbrace{N_c}_{\text{each loop contributes an extra } N_c \text{ for } N_c \gg 1.}$$

$$\uparrow$$

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{k^3 dk \cdot \overbrace{Vol_{S^3}}^{2\pi^2}}{(2\pi)^4} = \frac{1}{16\pi^2}\int k^2 d(k^2)$$

$$\Rightarrow R = R(\alpha_s(Q^2), 1)$$

$$\alpha_s \sim \frac{1}{\ln Q^2/\Lambda_{QCD}^2}$$

(check that this solves the above RGE; easiest by looking at RGE for α_s^{-1})

↑ comes from boundary condition of the diff. eq.

- We can trust all this only for large Q^2, where α_s is small. (e.g. for $Q^2 = m_Z^2$; $\alpha_s(m_Z) \approx 0.12$)
- $\Lambda_{QCD} \simeq 0.2$ GeV (depends strongly on scheme).
- The leading-order β-fct. is scheme-indep. (cf. our QED-derivation based only on discussion of divergences). So is the qualitative behaviour

$$\underbrace{\begin{array}{l} \alpha_s \to 0 \\ Q^2 \to \infty \end{array}}_{\text{"Asymptotic freedom"}}.$$

6.2 The 1-loop β-fct. of QCD

- To explicitly calculate the β-fct., it is more convenient not to focus on a specific observable but to renormalize the lagrangian. This includes renormalizing fields and the gauge-fixing parameter:
- $(A_\mu^a)^0 = Z_3^{1/2} A_\mu^a$; $\xi_0 = Z_3 \xi$

$$\Rightarrow (A_\mu)^0 \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{\xi_0}\right)\right) (A_\nu)^0$$

$$= Z_3 \, A_\mu \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{Z_3 \xi}\right)\right) A_\nu$$

$$= A_\mu \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{\xi}\right)\right) A_\nu \; - \underbrace{\delta Z_3}_{\equiv 1 - Z_3} \cdot \underbrace{A_\mu \left(k^2 \eta^{\mu\nu} - k^\mu k^\nu\right) A_\nu}_{\sim F^2}$$

at leading order

This is important since divergences in the gluon self-energy will be only of the "transverse form" $\sim k^2 \eta^{\mu\nu} - k^\mu k^\nu$.

- $\psi_0 = Z_2^{1/2} \psi$
- $c_0 = Z_c^{1/2} c$
- $m_0 = m - \delta m$
- $g_0 = Z_g \, g$

The above implies a counterterm (among others)

$$g \bar{\psi} \not{A} \psi \cdot \left(\underbrace{Z_2 Z_3^{1/2} Z_g}_{\equiv Z_{1,F}} - 1 \right)$$

(renormalization of $\bar{\psi} \not{A} \psi$-vertex),

i.e. $Z_g = Z_{1,F} \, Z_2^{-1} Z_3^{-1/2}$.

Thus, we can write $g = \underbrace{Z_{1,F}^{-1} Z_2 Z_3^{1/2}}\, g_0 = Z_g^{-1} g_0$

These are fixed by the counterterms needed to make the $\bar{\psi}\not{A}\psi$-vertex, the fermion-selfenergy & the gluon-selfenergy finite. (We will need to calculate them!)

<u>Note:</u> In QED, we had $e = Z_1^{-1} Z_2 Z_3^{1/2} e_0$ and $Z_1^{-1} = Z_2$. Thus, the photon-selfenergy was sufficient.

That's not true any more in QCD.

- As in QED, the leading order β-fct. is

$$\beta(g) = \frac{d}{d\ln\Lambda} g_0 = g \frac{d}{d\ln\Lambda}(Z_g).$$

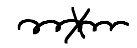
- Thus, all we need to do is to extract the coeff. of $1/\epsilon$ from $Z_{1,F}$, Z_2^{-1}, $Z_3^{-1/2}$, to add them & multiply by g.

① <u>Gluon self-energy</u> (Z_3)

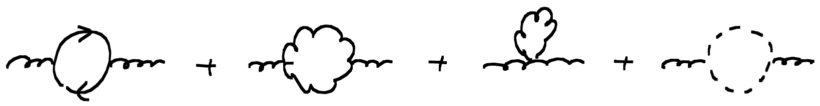
- The field redefinition $A_\mu \to A_\mu Z_3^{1/2}$ induces a counterterm

$$-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \cdot \delta Z_3 \,,$$

which gives rise to a Feynman rule

μ, a  ν, b $= i(-\eta^{\mu\nu} q^2 + q^\mu q^\nu)\, \delta^{ab}\, \delta Z_3$

- It has to cancel the divergence in

$i\Pi^{(1)}_{\mu\nu, ab} =$ 

a) b) c) d)

$$= i(q^2 \eta^{\mu\nu} - q^\mu q^\nu)\, \Pi_{(1)}(q^2)\, \delta^{ab}. \qquad \text{(as in QED)}$$

- In this notation: $\delta Z_3 = \Pi_{(1)}(0)\big|_{\frac{1}{\epsilon}\text{-term}}$

- Contribution a) is as in QED, up to the group-theoretic factor $T^a_{ij} T^b_{ji} = \mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. For N_f flavors, we get

$$\Pi^{a)}_{(1)} = -\frac{g^2 N_f}{12\pi^2 \epsilon} \qquad \text{(cf. QFT I)}$$

- It is useful to think about this a bit more generally:

The fermions could transform in any repres. (called r), generated by matrices $(T_r^a)_{ij}$. All that is required is that the T_r^a satisfy the same commut. relations as $T_F^a = T^a = \frac{\lambda^a}{2}$. We then have:

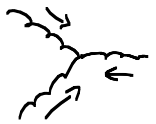
$$\mathrm{tr}\,(T_r^a T_r^b) = C(r)\,\delta^{ab}.$$

[group theoretically this is related to the "Dynkin index". One often sees $C(r) = T(r)$]

Our factor $\frac{1}{2}$ is actually $\frac{1}{2} = C(F)$.

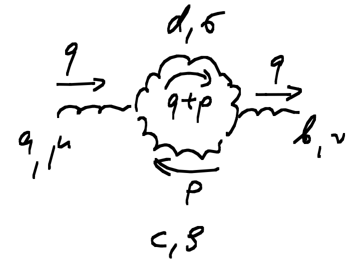
- Contribution b): (using Feynman gauge, $\xi = 1$)

Recall:



$$= -g f^{a_1 a_2 a_3} \left\{ (k_1 - k_2)_{\mu_3}\, \eta_{\mu_1 \mu_2} + \text{cyclic} \right\}$$

$\Rightarrow$



$$= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \cdot \frac{-i}{p^2} \cdot \frac{-i}{(p+q)^2}\, g^2 f^{acd} f^{bcd} \cdot$$

$$\cdot \left\{ (q-p)_\sigma \eta_{\rho\mu} + (p-(-q-p))_\mu \eta_{\rho\sigma} + ((-q-p)-q)_\rho\, \eta_{\mu\sigma} \right\} \cdot$$

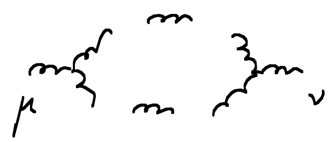
$$\uparrow\ \uparrow\ \uparrow \qquad \uparrow\ \uparrow\ \uparrow \qquad \uparrow\ \uparrow\ \uparrow$$
$$1\ \ 2\ \ 3 \qquad 2\ \ 3\ \ 1 \qquad 3\ \ 1\ \ 2$$

$$\cdot \left\{ (-q+p)^\sigma \eta_\nu{}^\rho + (-p-(q+p))_\nu\, \eta^{\sigma\rho} + (q+p-(-q))^\rho \eta_\nu{}^\sigma \right\}$$

$$\uparrow\ \uparrow\ \uparrow \qquad \uparrow\ \uparrow\ \uparrow \qquad \uparrow\ \uparrow\ \uparrow$$
$$1\ \ 2\ \ 3 \qquad 2\ \ 3\ \ 1 \qquad 3\ \ 1\ \ 2$$

- The prefactor $\frac{1}{2}$ is a symmetry factor:

We start with $\left(\frac{1}{2}\right)^2 \frac{1}{2} \cdot \frac{1}{2}$



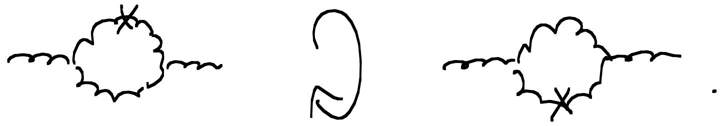
$\left(\frac{1}{2}\right)^2$: each prop.; $\frac{1}{2}$: 2 props.; $\frac{1}{2}$: two vertices (from Taylor expansion of $\exp(\dots)$)

- Note that, as opposed to $\lambda\varphi^3$-theory, the Feynman rule already incorporates the $3!$ different ways to act with $\frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\frac{\delta}{\delta j_3}$ on $AA\partial A$. (No explicit factor $3!$ appears.)

- The factors $\left(\frac{1}{2}\right)^2$, $\frac{1}{2}$, $\frac{1}{2}$ are supposed to cancel:

  - flip of two ends of each propagator
  - exchange of two propagators
  - exchange of two vertices.

- However, exchanging the upper & lower propagator is equivalent two rotating the diagram:



This is <u>not</u> a new way of connecting the lines. Hence, the corresponding factor $1/2$ is left over.

- One can prove that $f^{acd}f^{bcd} \sim \delta^{ab}$. In fact, $-if^{acd}$, interpreted as a set of $(N^2-1)\times(N^2-1)$ matrices, $(-if^a)^{cd}$, also generates a repres. of $SU(N)$. This repres. is known as the adjoint repres. (It corresponds to the Lie-alg. acting on itself via commutator). Thus:

$$f^{acd}f^{bcd} = (-if^a)^{cd}(-if^b)^{dc} = \mathrm{tr}\left(T_A^a T_A^b\right) = C(A)\,\delta^{ab}.$$

(We used total antisymmetry of f^{abc}.)

Alternatively, we can write

$$f^{acd} f^{bcd} = f^{dca} f^{dcb} = (-if^d)^{ac}(-if^d)^{cb} = (T^d T^d)_{ab}$$
$$= C_2(A)\,\delta_{ab}.$$

This last definition of "C_2" can also be made for any other representation :
$$(T_r^a T_r^a)_{ij} = C_2(r)\,\delta_{ij}.$$

$C_2(r)\cdot \mathbb{1}$ is the "quadratic Casimir operator" of the repres. "r".
(This name is then often also used for the constant $C_2(r)$.)

- We have just proved that $C_2(A) = C(A)$. More generally :
$$d(r)\,C_2(r) = d(A)\,C(r).$$
- One can show that, for $SU(N)$, $C_2(A) = C(A) = N$.

[For details see Peskin/Schröder]

- This is the group-theoretic factor of our contribution b).
- We now rewrite the denominator using the "Feynman parameter" as in QED:
$$\frac{1}{p^2}\cdot\frac{1}{(p+q)^2} = \int_0^1 dx\,\frac{1}{((1-x)p^2 + x\,(p+q)^2)^2} = \int_0^1 dx\,\frac{1}{(k^2-\Delta)^2},$$
with $k = p + xq$; $\Delta = -x(1-x)q^2$.
- We can replace $\int d^dp \to \int d^dk$.
- Let us now turn to the "numerator" (the two big curly brackets above, with open indices μ,ν).
- We can rewrite this structure in terms of k, q & x.

- Furthermore, since it appears in the integral

$$\int d^dk \, \frac{1}{(k^2-\Delta)^2} \; \cdots \; ,$$

any k-linear term gives zero after integration.

- Also: $\int d^dk \, f(k^2) \, k^\mu k^\nu = a \, \eta^{\mu\nu}$ by symmetry.

$$\Downarrow$$

$$\int d^dk \, f(k^2) \, k^2 = a \cdot d$$

$$\Downarrow$$

$$a = \int d^dk \, f(k^2) \, \frac{k^2}{d} \, .$$

- Thus, we can replace $k^\mu k^\nu \to \frac{k^2}{d} \eta^{\mu\nu}$.

- Doing all this, we find

$$\int \frac{d^dp}{(2\pi)^d} \cdot \frac{1}{p^2(p+q)^2} \cdot \{ \underset{\mu}{\cdots} \} \cdot \{ \underset{\nu}{\cdots} \} =$$

$$= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \, \frac{1}{(k^2-\Delta)^2} \cdot \Big\{ -\eta_{\mu\nu} k^2 \cdot 6 \cdot \left(1-\tfrac{1}{d}\right) - \eta_{\mu\nu} q^2 \left[(2-x)^2 + (1+x)^2 \right]$$
$$+ q_\mu q_\nu \left[(2-d)(1-2x)^2 + 2(1+x)(2-x) \right] \Big\}$$

(Beware! This and most of the following is just copied from Peskin/Schröder – errors are now less likely ☺)

We now Wick-rotate ($k_0 \to i k_0$) and use

$$\int \frac{d^dk}{(2\pi)^d} \cdot \frac{1}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \cdot \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n-\frac{d}{2}}$$

$$\& \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

(We already used the first integral for $n=2$ in QED. Here we also only need $n=2$. But we also need the second line, with the extra k^2.)

To prove the first line, rewrite (euclidean!!)

$$d^dk = \mathrm{Vol}(S^{d-1})\, k^{d-1} dk$$

and use

$$\int_0^\infty \frac{k^{d-1}dk}{(k^2+\Delta)^n} = \frac{1}{2}\int_0^\infty d(k^2) \frac{(k^2)^{\frac{d}{2}-1}}{(k^2+\Delta)^n}$$

$$= \frac{1}{2}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \int_0^1 dx\, x^{n-1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1}$$

$$(k^2+\Delta) \equiv \frac{\Delta}{x}, \quad \text{i.e. } d(k^2) = -\frac{\Delta}{x^2}dx \quad \& \quad k^2 = \Delta\cdot\left(\frac{1}{x}-1\right) = \frac{\Delta\cdot(1-x)}{x}$$

Then use

$$\int_0^1 dx\, x^{a-1}(1-x)^{b-1} \equiv B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Def. of the "Beta-fct." (of Mathematics, not to be confused with the β-fct. of QFT.)

important identity (can also be proved directly by doing the integral for integer a,b and analytically continuing)

The case with the extra k^2 in the numerator is also easy using $k^2 = (k^2+\Delta)-\Delta$.

Collecting all of this, we finally get:

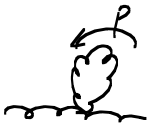
$$= \frac{i g^2}{(4\pi)^{d/2}} C_2(A)\, \delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}} \cdot$$

$$\cdot \Big\{ \Gamma\big(1-\tfrac{d}{2}\big) \eta^{\mu\nu} q^2 \big[\tfrac{3}{2}(d-1)\, x(1-x)\big] + \Gamma\big(2-\tfrac{d}{2}\big) \eta^{\mu\nu} q^2 \big[\tfrac{1}{2}(2-x)^2 + \tfrac{1}{2}(1+x)^2\big]$$

$$- \Gamma\big(2-\tfrac{d}{2}\big) q^\mu q^\nu \big[\big(1-\tfrac{d}{2}\big)(1-2x)^2 + (1+x)(2-x)\big] \Big\}$$

• Contribution c)

No new ideas, in fact much simpler than b)



$$= - g^2 C_2(A)\, \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \cdot \frac{1}{p^2}\, \eta^{\mu\nu} (d-1).$$

Note: This appears to be a quadratic divergence (like in  of $\lambda\varphi^4$-theory). We need to see that this divergence is cancelled by the other diagrams. On the other hand,

$$\int \frac{d^d p}{(p^2)^n} = 0 \quad \text{in dim. reg.}$$

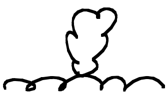
(Obvious from rescaling $p \to \alpha p$. Thus, in any axiomatic definition of dim. reg. which is consistent with the usual rules of integration, this identity must hold.)

This is not a contradiction: In fact, one has at

$$d = 2: \quad \int \frac{dp^2}{p^2} = \underset{\substack{\uparrow \\ \text{UV-pole}}}{\frac{1}{\epsilon}} - \underset{\substack{\uparrow \\ \text{IR-pole}}}{\frac{1}{\epsilon}} = 0$$

While it is thus completely consistent to simply set this diagram to zero, one can try to separate the IR/UV divergences, such that one can see that, in the end the latter cancel among the diagrams. This is done multiplying with $\frac{(q+p)^2}{(q+p)^2}$, introducing the Feynman parameter and working the algebra out as in b).

The result is

 $$= \frac{i g^2}{(4\pi)^{d/2}} C_2(A)\, \delta^{ab} \int_0^1 \frac{dx}{\Delta^{2-d/2}} \cdot \eta^{\mu\nu} q^2$$

$$\cdot \left\{ -\Gamma\left(1-\frac{d}{2}\right) \frac{1}{2} d(d-1) x(1-x) - \Gamma\left(2-\frac{d}{2}\right)(d-1)(1-x)^2 \right\}$$

Here you see the pole at $d=2$. Such a pole means a log.-divergence in $d=2$, i.e., a quadratic divergence (UV) at $d=4$.

Here is the pole (log.-divergence) at $d=4$.

- The IR-divergence now hides in $\int_0^1 \frac{dx}{\left[-q^2 x(1-x)\right]^{2-d/2}}$, which is $\sim \int_0 \frac{dx}{x}$ at $d=2$ (uncancelled in the second term, $\sim \Gamma(2-\frac{d}{2})$, since there is no numerator-factor x).

- Contribution d)

 $$= (-1) \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2} \frac{i}{(p+q)^2} g^2 f^{dac} f^{cbd} (p+q)^\mu p^\nu$$

$$= \cdots = \frac{ig^2}{(4\pi)^{d/2}} C_2(A)\,\delta^{ab} \int_0^1 \frac{dx}{\Delta^{2-d/2}} \,\cdot$$

$$\cdot \left\{ -\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu} q^2 \frac{1}{2} x(1-x) + \Gamma\left(2-\frac{d}{2}\right) q^\mu q^\nu x(1-x) \right\}.$$

- In the combination of b), c) & d), the coeff. of $\Gamma(1-\frac{d}{2})$ is $\sim (3d-3-d^2+d-1) = -2\left(1-\frac{d}{2}\right)(2-d)$

  $\uparrow$ pole $\frac{1}{1-\frac{d}{2}}$ of Γ-fct. is cancelled! No log-div. at $d=2$ $\Rightarrow$ No quadratic divergence at $d=4$.

- Using the freedom to replace $x \leftrightarrow (1-x)$ in any term in numerator, with some algebra, $\underbrace{\text{everything}}_{b),\,c)\,\&\,d)}$ can now be combined into:

$$\frac{ig^2}{(4\pi)^{d/2}} C_2(A)\,\delta^{ab} \int_0^1 dx\, \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right)\left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right].$$

- This can, in fact, be fully evaluated in terms of Γ-fcts. only, for generic d. (see Itzykson/Zuber, where however very little calculational detail is given).

- Extracting the pole is, however easy ($\Delta^{2-d/2} \to 1$ for $d \to 4$).

[Note: To get the finite term right, one has to expand $\Delta^{-\epsilon/2} \simeq 1 - \frac{\epsilon}{2} \ln(-q^2 x(1-x)) + \cdots$ before doing the x-integration.]

- In combination with $\Pi^{a)}_{(1)}$ (see above), we find:

$$\delta Z_3 = \Pi_{(1)}\Big|_{1/\epsilon\text{-part}} = -\frac{g^2 N_f}{6\pi^2 \epsilon}\cdot C(r) - \frac{g^2}{(4\pi)^2}\left(-\frac{5}{3}\right) C_2(A)\, \underbrace{\Gamma\!\left(2-\tfrac{d}{2}\right)}_{\sim \frac{1}{2-\frac{d}{2}} = \frac{2}{\epsilon}} + \cdots \Big|_{1/\epsilon\text{-part}}$$

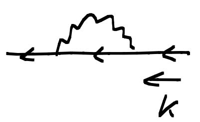
$$= -\frac{2g^2}{(4\pi)^2\epsilon}\left[\frac{4}{3}N_f\, C(r) - \frac{5}{3}\, C_2(A)\right]$$

if we replace $C(r)$ by 1, this is the QED vacuum-polariz. contribution, which is all there is in the QED β-fct.

here you already see the crucial opposite-sign effect. (to be enhanced to 11/3 by δZ_2 & $\delta Z_{1,F}$).

Z_2 (quark self-energy)

- δZ_2 has to cancel the divergence in



$$= (ig)^2 \int \frac{d^d p}{(2\pi)^d}\, \gamma^\mu T^a \frac{i}{\not p - \not k}\, \gamma_\mu T^a\, \frac{-i}{p^2}\,.$$

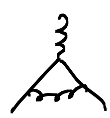
- This is a good excercise in loop-integrals & γ-matrix-algebra. (No new ideas are required.)

- Use $(T^a T^a)_{ij} = C_2(r)\,\delta_{ij}$.

- The result is $\delta Z_2 = -\frac{g^2}{(4\pi)^2}\cdot\frac{2}{\epsilon}\cdot C_2(r)$.

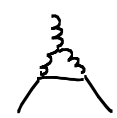
$Z_{1,F}$ (vertex)

The divergences of the two contributing diagrams



a)

+



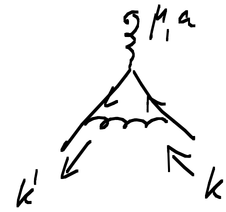
b)

have to be cancelled by the counterterm

$$(Z_{1,F}-1)\,\bar\psi\, ig\not{A}\,\psi \;=\; \delta Z_{1,F}\,\bar\psi\, ig\not{A}\,\psi$$

of the lagrangian.

- Contribution a)



$$= g^3\int\frac{d^dp}{(2\pi)^d}\; T^bT^aT^b\;\gamma^\nu\frac{1}{\not{p}+\not{k}'}\gamma^\mu\frac{1}{\not{p}+\not{k}}\gamma_\nu\cdot\frac{1}{p^2}$$

- Since there is no quadratic divergence which we would need to keep track of (unlike  etc.), it is sufficient to evaluate the diagram at $k = k' = 0$.

- The only new feature is in the group-theory factor:

$$T^bT^aT^b = T^bT^bT^a + T^b[T^a,T^b]$$
$$= C_2(r)\,T^a + \underbrace{iT^bf^{abc}T^c}_{if^{abc}\frac{1}{2}[T^b,T^c] \,=\, \frac{1}{2}\underbrace{if^{abc}\,if^{bcd}}_{-C(A)\delta^{ad} = -C_2(A)\delta^{ad}}T^d}$$
$$= \Big[C_2(r) - \tfrac{1}{2}C_2(A)\Big]\,T^a$$

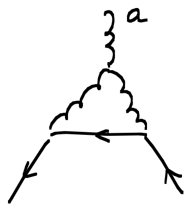
This is the color structure of the vertex at tree level.

- Use also $\displaystyle\int\frac{d^dp}{(2\pi)^d}\,\frac{\gamma^\nu\not{p}\gamma^\mu\not{p}\gamma_\nu}{(p^2)^3} = \int\frac{d^dp}{(2\pi)^d\, d}\cdot\frac{\gamma^\nu\gamma^\rho\gamma^\mu\gamma_\rho\gamma_\nu}{(p^2)^2}$

$$p^\rho p^\sigma \to p^2\,\frac{\eta^{\rho\sigma}}{d}\,.$$

- Doing the γ-algebra and extracting the pole is straightforward.

Contribution b)



$= \dots$ everything is very much as in a), just with a bit different explicit formulae...

- The most interesting part is again the color factor:

$$f^{abc} T^b T^c = \frac{1}{2} f^{abc}\, i f^{bcd} T^d = \frac{i}{2} C_2(A) T^a .$$

- Combining a) & b), one finds an overall $1/\epsilon$-divergence and the corresponding counterterm

$$\delta Z_{1,F} = -\frac{g^2}{(4\pi)^2} \cdot \frac{2}{\epsilon} \left(C_2(r) + C_2(A) \right) .$$

- Finally:

$$\beta(g) = g \cdot \left(-\frac{1}{2}\delta Z_3 - \delta Z_2 + \delta Z_{1,F} \right) \Big|_{\text{coeff. of } 1/\epsilon\text{-term}}$$

$$= \frac{g^3}{(4\pi)^2} \Big(\underbrace{\frac{4}{3} N_f C(r) - \frac{5}{3} C_2(A)}_{\text{from } Z_3} + \underbrace{2 C_2(r)}_{Z_2} - \underbrace{2 C_2(r) - 2 C_2(A)}_{Z_{1,F}} \Big)$$

$$= \frac{g^3}{(4\pi)^2} \left(\frac{4}{3} N_f C(r) - \frac{11}{3} C_2(A) \right)$$

Brief conceptual summary

renormaliz. scale M ; cutoff scale Λ

$$\beta(g_{phys.}) = \frac{d}{d \ln M} g_{phys.}(M)$$

$$g_{bare}(\Lambda) = Z_g(\Lambda / M)\, g_{phys.}(M)$$

$$\frac{d}{d\ln M}\, g_{phys.}(M) = \frac{d}{d\ln M}\, Z_g^{-1}(\Lambda/M)\, g_{bare}(\Lambda)$$

$$= -\frac{d}{d\ln\Lambda}\left(Z_g^{-1}(\Lambda/M)\right) g_{bare}(\Lambda)$$

$$= Z_g^{-1}(\Lambda/M)\,\frac{d}{d\ln\Lambda}\, g_{bare}(\Lambda) \underset{\text{1-loop}}{\approx} \frac{d}{d\ln\Lambda}\, g_{bare}(\Lambda)$$

$$\approx \frac{d\, Z_g(\Lambda/M)}{d\ln\Lambda}\; g_{phys.}(M)$$

Now in dim reg.: $M \to \mu$; $\frac{d}{d\ln\Lambda} Z_g = \left(\text{coeff. of } \frac{1}{\epsilon} \text{ in } \delta Z_g\right)$

Also: $Z_g = Z_{1,F}\, Z_2^{-1}\, Z_3^{-1/2} = (1+\delta Z_{1,F})(1-\delta Z_2)(1-\frac{1}{2}\delta Z_3)$

$$\approx 1 + (\delta Z_{1,F} - \delta Z_2 - \tfrac{1}{2}\delta Z_3)$$

Hence we just need to combine the coeffs. of the $\frac{1}{\epsilon}$-parts of these δZ's, as executed above.