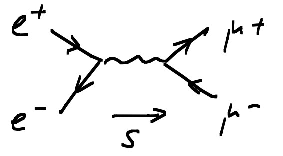
# 7 QCD – $e^+e^-$ to Hadrons & OPE

## 7.1 Tree-level result

- Recall our calculation of $e^+e^- \to \mu^+\mu^-$ in QFT.

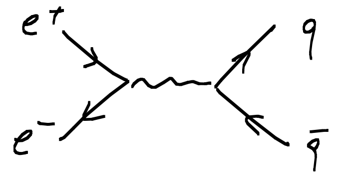



$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- Total cross sect.: $\int d\Omega \cdots = \int d\varphi \sin\theta \, d\theta \cdots = 2\pi \int_{-1}^{1} d\cos\theta \cdots$

$$\Rightarrow \quad \sigma = 2\pi \frac{\alpha^2}{4s} \int_{-1}^{1} dx\,(1+x^2) = \frac{\pi\alpha^2}{2s}\left(x+\frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{\pi\alpha^2}{2s}\cdot\frac{8}{3} = \frac{4\pi\alpha^2}{3s}$$

- For $e^+e^- \to$ hadrons, we have at leading order




with $q = (u, d, s, c, b, t)$

(to be summed over to the extent that the quarks are not too heavy; at the lowest energies where PQCD is valid this would be just $u, d, s$.)

- The relevant quark-photon-vertex comes from

$$\mathcal{L} \supset \sum_f \bar{q}_f (i\not{D} - m_f) q_f \supset \sum_f \bar{q}_f (-e e_f \not{A}_\gamma) q_f$$

contains $U(1)$, $SU(2)$ & $SU(3)$ gauge fields

vector in $SU_3$-fundamental repres., i.e.

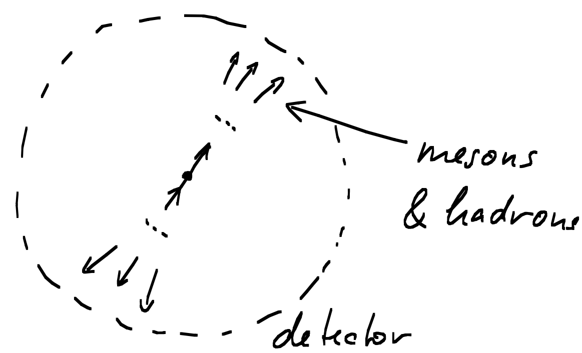
$$q_f \to q_{f,i} \; ; \; i = 1,2,3$$

$$\Rightarrow \quad \sigma_{e^+e^- \to q\bar{q}} = \sigma_{e^+e^- \to \mu^+\mu^-} \cdot 3 \sum_f e_f^2$$

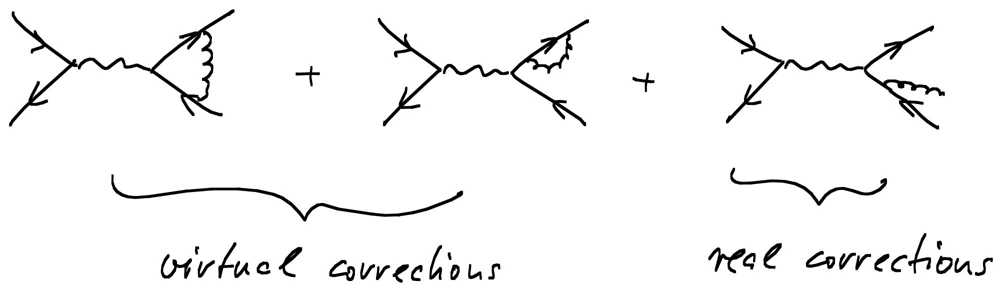
or $$\sigma_{e^+e^- \to q\bar{q}} = 3 \cdot \frac{4\pi\alpha^2}{3s} \cdot \left(\sum_f e_f^2\right) \qquad \left(e_{u,c,t} = \frac{2}{3}\,; \; e_{d,s,b} = -\frac{1}{3}\right)$$

This was historically an important step towards establishing QCD.

Note: All of the above also works if one stays "differential" in $\theta, \varphi$. The corresponding observable is the angular distribution of two jets:




## 7.2 NLO-result (in this case leading-order in $\alpha_s$)

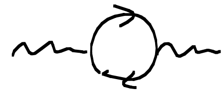



(We ignore NLO corrections $\sim \alpha$ since $\alpha \ll \alpha_s$)

The correction to the total cross sect. based on these diagrams turns out to be finite for a number of non-trivial reasons:

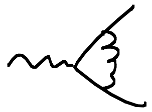
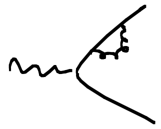
## Ultraviolet:

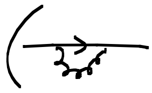
Any UV-divergence would have to be absorbed in $e$ (since this is the only coupling constant in the LO result). However, we already know that the LO-renormalization of $e$ (or $\alpha$) follows from



sum over all relevant charged particles.

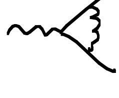
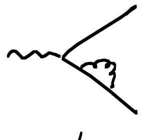
Hence, no renormalization $\sim \alpha_s$ can arise. Thus, the UV divergence cancel among

 ;  & $Z_q^{-1/2}$ for each leg (cf. LSZ)

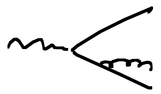
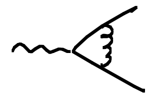
$\sim -\frac{1}{2}\Big($  $\Big)$ .

<u>Note:</u> As we learned, we can also calculate the cross sect. from 1PI-diagrams only and include a correction $\sim Z_q^{1/2}$ for each ext. leg.

## Infrared & Colinear

- The diagrams  &  exhibit divergences from the loop-integration region where the gluon momentum is

a) small – (infrared)
b) almost parallel to (one of) the quark momenta – (colinear)

- The diagram  exhibits divergences (from phase-space integration for soft & colinear gluons.
- All of this can be treated in dim. reg. (phase-space in $d$ dims.!).
- Poles up to $\frac{1}{\epsilon^2}$ arise since in  the gluon can be soft & colinear at the same time.
- In the end, all IR/colinear poles cancel among themselves (to see this explicitly, one needs to use

 $= 0 = c \cdot \left( \frac{1}{\epsilon_{uv}} - \frac{1}{\epsilon_{IR}} \right)$

$\uparrow$

$k^2 = 0$ (for $m_q = 0$),

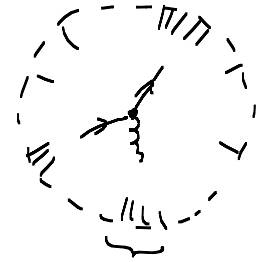
as we already did in the last chapter).

- A systematic proof of finiteness to all orders (in QED) is known as the Kinoshita-Lee-Nauenberg theorem (cf. Weinberg's book).
- The total correction of $\sigma_{e^+e^- \to \text{hadrons}}$ is finite & very simple:

$$\sigma_{e^+e^- \to \text{hadrons}} = \sigma_{e^+e^- \to q\bar{q}} \cdot \left(1 + \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right)$$

(cf. Book by Ellis/Stirling/Webber & my notes on "Pert. QCD" in ITP library ; <u>also</u>: Peskin/Schröder)

- If one wants to stay at the "differential" level in $\theta, \varphi$, one has to deal with "jet definitions" (it is important to use so called "IR-safe" jet definitions:



jet in detector

(a "jet definition" means a procedure to reassemble the meson/hadron momenta into two, three or more "jet momenta".)

- The observation of "3 jet events" were one of the final steps in establishing QCD (PETRA-collider at DESY).

## 7.3 Operator product expansion (OPE)

For this particular observable, a "cleaner" treatment based on the OPE is available:

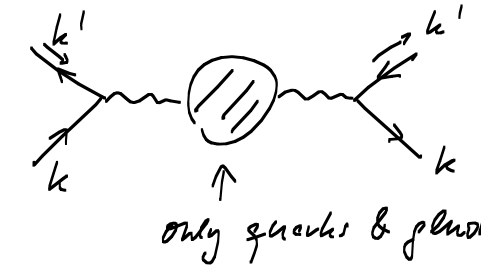
- We first appeal to the "optical theorem" known from QM. Its QFT-version reads (applied to $e^+e^- \to$ hadrons)

$$\sigma_{tot}(e^+e^- \to \text{hadr.}) = \frac{1}{2s} \operatorname{Im} \mathcal{M}(e^+e^- \to e^+e^-)$$

use only intermediate hadronic states

(Idea of proof: $$1 = (1 + iT)(1+iT)^+$$
$$= 1 + \underbrace{i2\operatorname{Im} T}_{\operatorname{Im}\mathcal{M}} + \underbrace{TT^+}_{\sigma}$$ )

diagrammatic:




- explicitly: $i\mathcal{M} = (-ie)^2\, \bar{u}(k)\gamma_\mu v(k')\, \frac{-i}{s}\, \left(i\Pi_h^{\mu\nu}(q)\right) \frac{-i}{s}\, \bar{v}(k)\gamma_\nu u(k)$

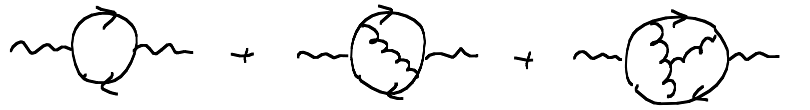
hadronic part of vacuum polarization of photon

(Working this out, one finds

$$\sigma(e^+e^- \to \text{hadr.}) = -\frac{4\pi\alpha}{s}\, \mathrm{Im}\,\Pi_h(s) \quad ; \quad s = (k+k')^2$$

$$\Pi_h^{\mu\nu}(q) \equiv \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi_h(q^2)\,.)$$

- In terms of Feynman diagrams, $\Pi_h^{\mu\nu}$ corresponds to



etc.

- Since the first coupling of the photon is always to quarks (via the electromagnetic current $j^\mu = \sum_f e_f\, \bar{q}_f \gamma^\mu q_f$ ), we have

$$i\Pi_h^{\mu\nu}(q) = -e^2 \underbrace{\int d^4x\, e^{iqx}}\, \langle 0|\, T j^\mu(x) j^\nu(0)\, |0\rangle$$

This comes from the two Fourier transforms & the mom.-conserving $\delta$-fct. in $\Pi$.

- To treat this further, we appeal to the more general & important fact (with many applications beyond QCD), known as the

## Operator Product Expansion (for 2 local operators $A(x)$ & $B(y)$)

$$\boxed{A(x)B(y) \sim \sum_C F_C^{AB}(x-y)\cdot C(y)}$$

(Wilson, 1969)

Sum over set of local operators $C$

fcts. $F_C^{AB}$ of the separation $x-y$ of the operators $A$ & $B$. (in general complex & singular for $x-y \to 0$)

- The symbol "$\sim$" means equality up to non-singular terms, i.e.

$$f(x,y) \sim g(x,y) \iff f(x,y) - g(x,y) \text{ is analytic at the point } x=y.$$

- In this sense, the OPE can viewed as a Laurent series expansion of $A(x)B(y)$ with operator-valued coefficients (only qualitatively).
- Dimensional analysis $\Rightarrow$

$$F_C^{AB}(x-y) \approx \begin{pmatrix}\text{numerical}\\ \text{constant}\\ \text{depending}\\ \text{on } A,B,C\end{pmatrix}\cdot(x-y)^{[C]-[A]-[B]}$$

This is only true for $(x-y)\to 0$ as long as the power of $(x-y)$ is negative.

mass dimensions of $A, B, C$

- Note: In the type of QFT we discuss here, there is a distinguished type of operators associated with fields of the underlying class. Lagrangian ($\varphi$, $A_\mu$, $\psi$ etc.). We think of $A, B, C$ as built from such fields. It is then clear that $[C]$ grows with growing "complexity" of $C$. The most singular terms are associated with the simplest $C$s. This is one of the points making the OPE so useful.

- Note also: There are QFTs (like the 2d CFT describing the worldsheet dynamics of string theory) where "fundamental fields" play a less central role. In fact, the OPE is even more important in such theories (but we will not discuss this).

- The OPE can be formulated more precisely and proven (cf. e.g. Weinberg II; Itzykson/Zuber; etc.).

- The fact that $A(x)B(y)$ is singular at $x=y$ is a generic feature of QFTs. As the simplest example, consider $\varphi(x)\varphi(y)$ in a scalar theory, where we know that

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik(x-y)}}{k^2-m^2+i\varepsilon}\,,$$

which is clearly singular at $x-y=0$.

- <u>Warning:</u> A QFT is only well-defined after renormalization, which requires a renormalization scale (such as the "$\mu$" of the MS schemes). One finds a log. dependence on $\mu$, which in the OPE leads to corrections

$$\frac{1}{(x-y)^\alpha} \to \frac{1}{(x-y)^\alpha}\,\ln\left((x-y)^2\mu^2\right)$$

of the singular behaviour of the $F_C^{AB}$'s.

- We now return to QCD application, where we need $j_\mu(x)j_\nu(0)$. Since we only need $\langle 0|\cdots|0\rangle$ of the OPE, we restrict $\sum_C$ to gauge-inv. Lorentz scalars. The simplest such operators are $\mathbb{1}$; $\bar{q}q$; $\mathrm{tr}\,F^2$ (with lowest mass dim.!)

$$j_\mu(x)j_\nu(0) \sim C^{\mathbb{1}}_{\mu\nu}(x)\cdot\mathbb{1} + C^{\bar{q}q}_{\mu\nu}\cdot\bar{q}q(0) + C^{F^2}_{\mu\nu}\cdot\mathrm{tr}\,F^2(0)$$

- QCD has a "chiral symmetry": $q \to e^{i\gamma^5\alpha} q$ in the massless limit.

$$[ \quad \bar{q}\gamma^\mu q \longrightarrow (e^{i\gamma^5\alpha}q)^+\gamma^0\gamma^\mu e^{i\gamma^5\alpha}q = q^+e^{-i\gamma^5\alpha}\gamma^0\gamma^\mu e^{i\gamma^5\alpha}q$$
$$= \bar{q}\gamma^\mu q\,,$$

while

$$\bar{q}q \longrightarrow q^+e^{-i\gamma^5\alpha}\gamma^0 e^{i\gamma^5\alpha} = \bar{q}e^{2i\gamma^5\alpha}q \neq \bar{q}q \quad ]$$

- Hence $\bar{q}q$ can appear in the OPE of $j_\mu(x)j_\nu(0)$ only if $m \neq 0$.
- Thus, we have on dimensional grounds

$$C^{1}_{\mu\nu} \propto x^{-6}\;;\quad C^{\bar{q}q}_{\mu\nu} \propto m x^{-2}\;;\quad C^{F^2}_{\mu\nu} \propto x^{-2}$$

prop. to $x^{-6}$ for $x \to 0$.

- If we Fourier transform and use the conservation of the el.-mag. current, we find

$$-e^2\int d^4x\, e^{iqx} j_\mu(x) j_\nu(0)$$
$$\sim -ie^2(q^2\eta_{\mu\nu} - q_\mu q_\nu)\left\{C^1(q^2)\mathbb{1} + C^{\bar{q}q}(q^2)\, m\,\bar{q}q(0) + C^{F^2}(q^2)\,\mathrm{tr}F^2(0) + \dots\right\}$$

with

$$C^1 \propto 1\;;\quad C^{\bar{q}q} \propto C^{F^2} \propto \frac{1}{(q^2)^2} \quad \text{at large } q^2 \text{ by dim. analysis.}$$

This directly gives us $\Pi_h(q^2)$:

$$\Pi_h(q^2) \sim -e^2\, C^1(q^2) + C^{\bar{q}q}(q^2)\, m \underbrace{\langle\bar{q}q\rangle} + C^{F^2}(q^2)\langle \mathrm{tr}F^2\rangle + \dots$$

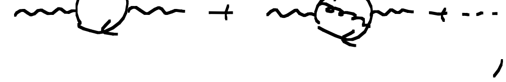
$\Downarrow$

($\sigma$ from opt. theorem)

[ <u>Note</u>: This is non-zero even at $m = 0$ since chir. symm. is "spont. broken",

i.e., $|0\rangle$ is not invariant although $\hat{H}$ is invariant.
It was nevertheless ok to use the chiral-symm. argument above since the OPE respects even symmetries which are spont. broken ($\to$ Weinberg II).

- The coefficients of the OPE can be calculated in pert. theory. To understand this, let us first assume that we us the <u>perturbative</u> vacuum (i.e. no quarks or gluons) in $\langle 0| \cdots |0\rangle$. Then only $\mathbb{1}$ contributes, and we have

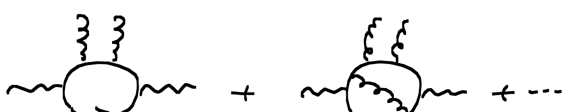
$$\sigma(e^+e^- \to hadr) \sim \mathrm{Im}\, c^{\mathbb{1}}(q^2) \sim$$  ,

which fixes $c^{\mathbb{1}}(q^2)$ in terms of a pert. calculation (this corresponds to the naive parton model calculation we did earlier).

- To fix $c^{\bar{q}q}(q^2)$, we use $|1\text{-quark}\rangle$ instead of $|0\rangle$ to "sandwich" our $j$-$j$-OPE

$$\Rightarrow \quad c^{\bar{q}q}(q^2) \sim$$ 

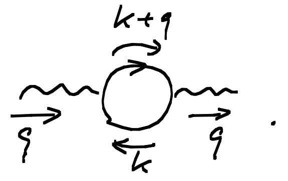
- Analogously, we have

$$\Rightarrow \quad c^{F^2}(q^2) \sim$$  .

<u>Note:</u> In the application to the "real world" $e^+e^- \to$ hadrons, we need to use the true (non-perturbative!) vacuum state of QCD to sandwich the $j$-$j$ OPE. The contribution of $c^{\bar{q}q}$, $c^{F^2}$ etc. is then non-zero. One can think of the ext. quark and gluon lines in the above diagrams as connecting to quarks & gluons present in this vacuum.

- A further problem arises if we think about the relevant values of $q^2$:

Focus on the simplest diagram:
(However, the argument can be made more general.)




- Ignoring for simplicity the numerator-structure, the relevant integral is of the type $\int d^4k \, \frac{1}{(k^2+i\varepsilon)((k+q)^2+i\varepsilon)}$

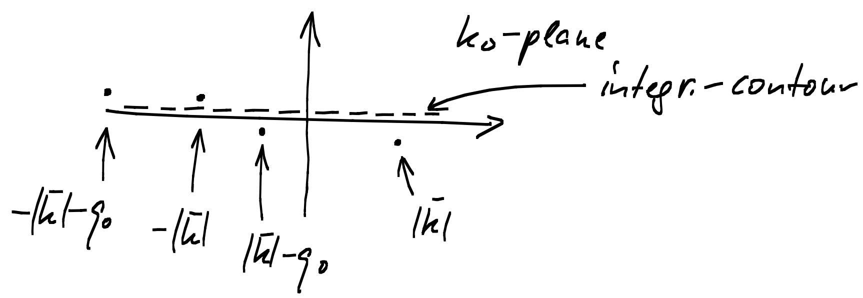
- Let first $q^2 > 0$ and choose a frame $q = (q_0, \bar{0})$:

1st propagator $\Rightarrow$ poles at $k_0 = \pm\sqrt{\bar{k}^2 - i\varepsilon} \approx \pm(|\bar{k}| - i\varepsilon')$

2nd propagator $\Rightarrow$ poles at $k_0 + q_0 = \pm(|\bar{k}| - i\varepsilon')$, i.e.

$$k_0 = -q_0 \pm (|\bar{k}| - i\varepsilon')$$

- For sufficiently small $\bar{k}$, the $k_0$-integration contour is "pinched" between the poles:




$\Rightarrow$ Wick rotation impossible. On-shell region important.

- Now let $q^2 < 0$ and choose frame where $q = (0, \bar{q})$.

1st prop. $\Rightarrow$ poles as before

2nd prop. $\Rightarrow$ poles at $k_0 = \pm(|\bar{k} + \bar{q}| - i\varepsilon')$

$\Rightarrow$ 1st & 3rd quadrant free of poles; Wick rotation possible; the resulting euclidean integral is dominated by scale $Q^2 = -q^2$ and pert. theory is justified at $Q^2 \gg \Lambda_{QCD}^2$.

So let us now assume that we can evaluate $\Pi_h(q^2)$ at $q^2$ large & negative using PQCD + $\langle 0|\bar{q}q|0\rangle$, $\langle 0|\mathrm{tr}F^2|0\rangle$ etc.
How does that help us to get $\sigma(q^2)$ for $q^2$ large & positive?

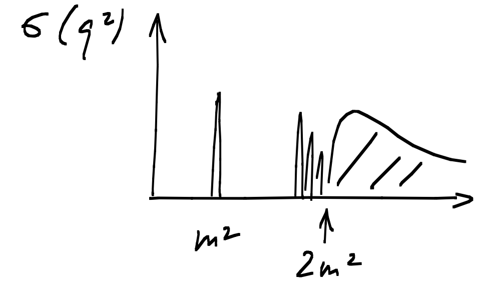
- To understand this, recall that for the real scalar field we had (cf. Sects. 4 & 6 of QFT I):

$$\langle T\varphi(x)\varphi(y)\rangle = \int_0^\infty dm'^2 \, D(x-y, m'^2)\, \sigma(m'^2)$$

or

$$\frac{1}{q^2-m_0^2-\Pi(q^2)} = \int_0^\infty dm'^2 \, \frac{1}{q^2-m'^2+i\varepsilon}\, \sigma(m'^2)$$

with

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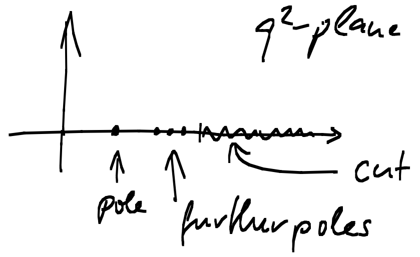
- We immediately see that the r.h. side is analytical except for $q^2$ real and positive:

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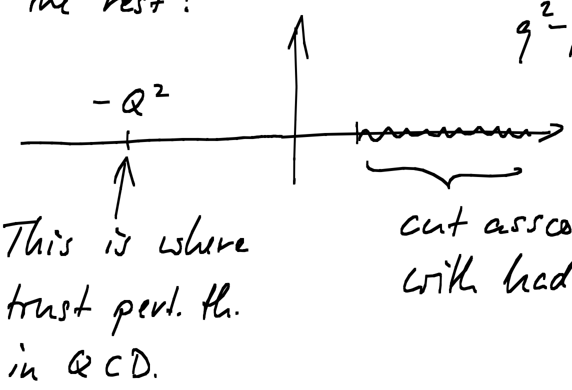
- All of this applies to $\langle T j j\rangle$ and hence to $\Pi_h(q^2)$, except that we now have no clean separation between the first pole and the rest:

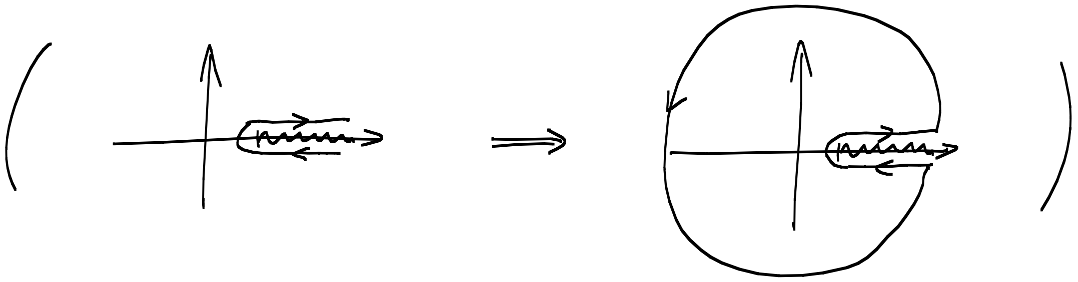



(analyt. structure of $\Pi_h(q^2)$)

- Let us now calculate

$$I_n \equiv \int_0^\infty ds \, \frac{s\sigma(s)}{(s+Q^2)^{n+1}} = -4\pi\alpha \int \frac{dq^2}{2\pi i} \cdot \frac{1}{(q^2+Q^2)^{n+1}} \cdot 2i\, \mathrm{Im}\, \Pi_h(q^2)$$

$$= -4\pi\alpha \int \frac{dq^2}{2\pi i} \cdot \frac{1}{(q^2+Q)^{n+1}} \; \text{Discontinuity}\left\{ \Pi_n(q^2) \right\}$$



$$= -4\pi\alpha \oint \frac{dq^2}{2\pi i} \cdot \frac{1}{(q^2+Q^2)^{n+1}} \cdot \Pi_n(q^2)$$

$$= -4\pi\alpha \; \text{Residue of} \left\{ \frac{\Pi_n(q^2)}{(q^2+Q^2)^{n+1}} \right\} \text{ at } q^2 = -Q^2$$

$$\boxed{\Pi_n(q^2) = \Pi_n(-Q^2) + (q^2+Q^2) \frac{d}{dq^2} \Pi_n(q^2)\Big|_{q^2=-Q^2} + \cdots}$$

$$= \frac{-4\pi\alpha}{n!} \left(\frac{d}{dq^2}\right)^n \Pi_n(q^2) \Big|_{q^2=-Q^2}$$

To this, we can apply pert.-th. + OPE, giving

$$\int_0^\infty ds \frac{s\,\sigma(s)}{(s+Q^2)^{n+1}} = \frac{4\pi\alpha^2}{n(Q^2)^n} \cdot \sum_f e_f^2 + O(\alpha_s(Q^2)) + O\left(\frac{1}{(Q^2)^2}\right)$$

"ITEP sum rule"
(Voloshin, Vainshtein, Novikov, Shifman, Zakharov ~ '78)

This is just the "moment" of our leading-order result, which we can also get from

Im (  )

pert. corrections

power corrections, associated with higher terms in OPE

Getting $\sigma(s)$ from all $I_n$'s is non-trivial ...