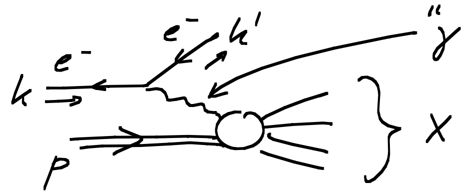
8 Parton Distribution Functions (PDFs)

8.1 Deep inelastic scattering (DIS)

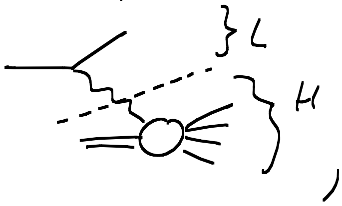
- Even more important than $e^+e^- \to$ hadr. are processes with hadrons in the initial state (e.g. for LHC). The simplest example is DIS:



"γ^*" with mom. $q = k - k'$

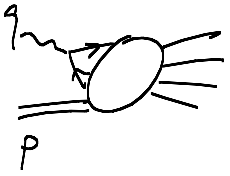
$(Q^2 \equiv -q^2)$

- We can split the amplitude in leptonic & hadronic part

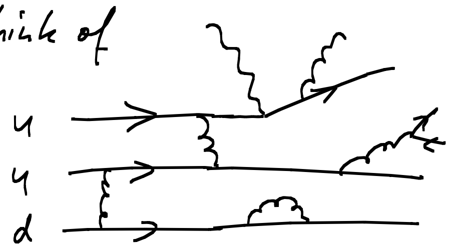


,

where $H^\mu =$



(Think of



etc.)

$$= \int d^4x\, e^{-iqx} \langle X | \bar\psi(x) (-iee_f \gamma^\mu) \psi(x) | p \rangle$$

$$= (2\pi)^4 \delta^4(p + q - p_X) \langle X | j^\mu(0) | p \rangle \equiv (2\pi)^4 \delta^4(\cdots) H_T^\mu$$

$$j^\mu(0) = -iee_f \bar\psi(0) \gamma^\mu \psi(0)$$

and

$$L_\mu = \bar u(k') (-ie\gamma_\mu) u(k) \frac{-i}{q^2} .$$

We find:

$$d\sigma^{ep \to eX} = \frac{1}{2s} (2\pi)^4 \delta^4(k' + p_X - k - p) |T|^2 \frac{d^3k'}{2k_0' (2\pi)^3} \cdot \underbrace{\text{"}dX\text{"}}_{\equiv \sum_X}$$

$$= \frac{1}{2s} \underbrace{\sum_X (2\pi)^4 \delta^4(\ldots)\, H_T^\mu H_T^{\nu *}}_{\sim W^{\mu\nu}} \underbrace{L_\mu L_\nu^*}_{\sim L_{\mu\nu}} \frac{d^3k'}{(2\pi)^3 2k_0'} \,.$$

$$\Rightarrow d\sigma \sim L_{\mu\nu} W^{\mu\nu} \quad \text{with} \quad L_{\mu\nu} \equiv e^2 \, tr(k'\gamma_\mu k \gamma_\nu)$$

and $W_{\mu\nu} \equiv \frac{1}{4\pi} \sum_X \langle p | j_\mu^+(0) | X \rangle \langle X | j_\nu(0) | p \rangle (2\pi)^4 \delta^4(q+p-p_X)$

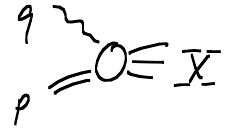
$\downarrow$ prove this!

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4x \, e^{iqx} \langle p | j_\mu^+(x) j_\nu(0) | p \rangle$$

[<u>Note:</u> Here, we could again use OPE (in a slightly more complicated way than before), but we will not do so]

We will evaluate $W_{\mu\nu}$ in (naive) perturbation theory (with partons in the initial state) and reintroduce non-pert. effects by carefully analysing its divergence structure. This is more intuitive than the OPE approach.

In doing so, it is convenient to define a $\gamma^* p$ total cross section:



$$; \quad \sigma^{tot} = \int \frac{1}{2\underbrace{(W^2+Q^2)}} |T|^2 (2\pi)^4 \delta^4(\ldots)\, dX$$

(Some authors use $W^2 + Q^2 \to W^2$.)

(This is a pure definition or convention since there is no well-defined "flux" of γ^*s. It is, however, "approx." meaningful at $W^2 \gg Q^2$ by the uncertainty principle.)

Leaving the photon-index open, we have $\sigma^{tot}_{\mu\nu}$ by an analogous definition. Obviously, $W_{\mu\nu} \sim \sigma^{tot}_{\mu\nu}$.

$\sigma^{tot}_{\mu\nu}$ can be calculated from σ^{tot}_T & σ^{tot}_L (together with $q^\mu \sigma^T_{\mu\nu} = 0$ & $q^\nu \sigma^T_{\mu\nu} = 0$). We just state the results:

$$\frac{d\sigma^{ep\to eX}}{dx\, dQ^2} = \frac{4\pi\alpha_{em}^2}{xQ^4}\left\{ (1-y+y^2/2)\, F_2(x,Q^2) - \frac{y^2}{2} F_L(x,Q^2) \right\}$$

$$x = Q^2/(W^2+Q^2) \; ; \; y = (W^2+Q^2)/\underset{\uparrow \atop (k+p)^2}{s}$$

$$F_2 = F_T + F_L$$

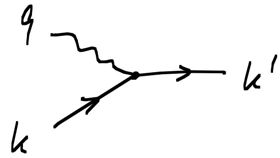
$$F_{T,L} = \frac{Q^2}{\pi e^2}\sigma_T$$

(F_T, F_L (or F_2, $F_1 \equiv \frac{F_2 - F_L}{2x}$) are "structure fcts.", which are useful since they don't depend on y. Otherwise, they are mainly of historical interest.)

We will now focus on calculating $\sigma_{T,L}$.

8.2 DIS on partons (LO)

(cf. Book of R. Fields; our $\sigma_{T,L}$ are in his conventions.)



(forget k & k' used earlier!)

Let $k = yP$ (forget the y-variable used earlier!)

$$\hat{\sigma}^{\gamma^* q}_T = \int \frac{1}{2y(W^2+Q^2)} |T|^2 (2\pi)^4 \delta^4(k'-k-q) \frac{d^3k'}{(2\pi)^3 2k_0'}$$

"^" for partonic

extra factor reducing the cms-energy appropriately.

One easily finds: (it's convenient to calculate

$$\underset{\substack{\uparrow \\ \sim\, \eta^{\mu\nu}\sigma_{\mu\nu}}}{\sigma^{\varepsilon}} = 2\sigma_T - \sigma_L \ \& \ \sigma_L .)$$

$$\hat{\sigma}_L = 0 \ ; \ \hat{\sigma}_T = \frac{\pi e^2}{Q^2}\,\delta(1 - x/y) \qquad (e_f \equiv 1 \text{ for simplicity})$$

$$\Rightarrow \boxed{\sigma_{T,L} = \int_0^1 dy\, \hat{\sigma}_{T,L}(y)\, q(y)}$$

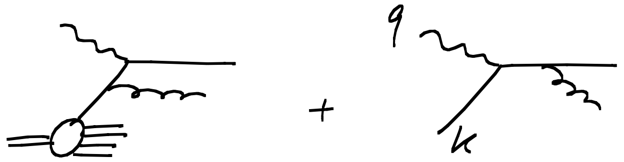
Here we are making the crucial assumption that the proton can be treated as a beam of independent partons (quarks), on which γ^* scatters incoherently.

$$\Rightarrow F_L = 0 \ ; \ F_2 = F_T = x \cdot q(x)$$

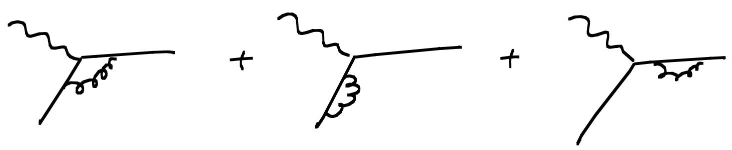
$$\left(F_2(x, Q^2) = F_2(x) \ - \ \text{"Bjorken scaling", 1969} \right)$$

8.3 DIS on partons (NLO) & Altarelli-Parisi-eq.

- real corrections:



- virtual corrections:



$$k = y \cdot p \ ; \ z \equiv x/y \ ; \ \hat{s} \equiv (yp + q)^2$$

$$(\hat{s} = 0 \Leftrightarrow z = 1)$$

- We calculate in $d = 4 + \epsilon$ dims.

- Tree level: $\hat{\sigma}_\Sigma^{(0)} = \sigma_0\, \delta(1-z)$ with $\sigma_0 = \dfrac{2\pi e^2}{Q^2}$
  (see above)

  In principle, we need to use the expression $\sigma_{0,d}$ in d dims. However, σ_0 will in the end multiply a finite quantity and its ϵ-terms will be irrelevant.

- Real correction:

$$d\hat{\sigma}_\Sigma^{real} = \frac{1}{2y(W^2+Q^2)}\, |T|^2\, \underbrace{dX_d^{(2)}}_{\text{2-particle phase space in } d \text{ dims.}}$$

$$\hat{\sigma}_\Sigma^{real} = \sigma_0\, \frac{2\alpha_s}{3\pi} \left(\frac{(1-z)Q^2}{z\, 4\pi\mu^2}\right)^{\epsilon/2} \frac{\Gamma(1+\epsilon/2)}{\Gamma(1+\epsilon)} \cdot z \cdot \left\{ \frac{1+z^2}{1-z}\cdot\frac{2}{\epsilon} - \frac{3}{2}\cdot\frac{1}{1-z} - z + 3 + \cdots \right\}$$

$$\hat{\sigma}_\Sigma^{virtual} = \sigma_0\, \delta(1-z)\, \frac{2\alpha_s}{3\pi} \left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2} \frac{\Gamma(1-\epsilon/2)\,\Gamma^2(1+\frac{\epsilon}{2})}{\Gamma(1+\epsilon)} \cdot z \cdot \left\{ -\frac{8}{\epsilon^2} + \frac{6}{\epsilon} - 8 + \cdots \right\}$$

Together:

$$\sigma_\Sigma = \int_x^1 dy \left(\hat{\sigma}_\Sigma^{(0)} + \hat{\sigma}_\Sigma^{real} + \hat{\sigma}_\Sigma^{virtual} \right) \cdot q(y)$$

Use $z = x/y$ as integr. variable and consider $F_\Sigma = \frac{1}{2} \cdot \frac{Q^2}{\pi e^2}\, \sigma_\Sigma$ to save some writing and to work with a dim.-less quantity.

- Focussing only on the double-pole $1/\epsilon^2$, we find:

$$F_\Sigma(x,Q^2) = x\int_x^1 \frac{dz}{z}\cdot\frac{2\alpha_s}{3\pi}\left\{\frac{1+z^2}{1-z}\,\frac{2}{\epsilon}\cdot(1-z)^{\epsilon/2} + \delta(1-z)\left(-\frac{8}{\epsilon^2}\right)\right\} q(x/z)$$

Note: The first term also gives a double pole because of the divergence at $z=1$:

$$\frac{2}{\epsilon}\int_x^1 \frac{dz}{z}\cdot\frac{1+z^2}{1-z}(1-z)^{\epsilon/2}\,q\left(\frac{x}{z}\right) = q(x)\,\frac{8}{\epsilon^2}+\cdots$$

This cancels the explicit double pole!

- To extract the $1/\epsilon$-part, we need to look at this cancellation in more detail. Consider the fct.

$$\frac{1}{(1-z)^{1-\epsilon/2}} - \frac{2}{\epsilon}\,\delta(1-z) \equiv f_\epsilon(z).$$

- $f(z) \equiv \lim\limits_{\epsilon\to 0} f_\epsilon(z)$ can be understood as a distribution:

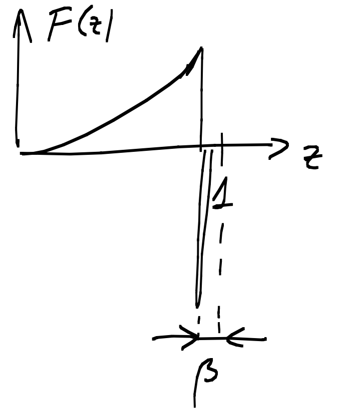
① $f(z) = \frac{1}{1-z}$ for any $z<1$

② $\int_0^1 dz\, f(z)\,g(z) = \lim\limits_{\epsilon\to 0}\left(\int_0^1 dz\, f_\epsilon(z)\,g(z)\right) = \cdots = \int_0^1 \frac{g(z)-g(1)}{1-z}$,

in particular $\int_0^1 dz\, f(z) = 0$.

- Such a fct. is called a "+"- fct., $f(z) \equiv \frac{1}{(1-z)_+}$, since it can be realized by adding a δ-fct. at $z=1$ to ensure $\int_0^1 dz\, f(z) = 0$:

$$F(z)_+ \equiv \lim_{\beta\to 0}\left\{F(z)\,\theta(1-\beta-z) - \delta(1-\beta-z)\int_0^{1-\beta} F(y)\,dy\right\}$$



& $\beta \to 0$.

- Now, putting everything together and expanding

$$\left(\frac{Q^2}{\mu^2}\right)^{\epsilon/2} = 1 + \frac{\epsilon}{2} \ln Q^2/\mu^2 + \cdots \quad , \text{ we get:}$$

$$F_\Sigma(x, Q^2) = x \int_x^1 \frac{dz}{z}\, q(y) \left[\delta(1-z) + \frac{\alpha_s}{2\pi}\left(P(z)\left(\frac{2}{\epsilon} + \ln\frac{Q^2}{\mu^2}\right) + \underset{\substack{\uparrow \\ \text{finite}}}{\cdots}\right)\right]$$

with the "splitting fct."

$$P(z) = \frac{4}{3}\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

- To get rid of the poles, we renormalize the quark distribution:

$$q(x) \to q_0(x) = q_{phys.}(x) - \frac{\alpha_s}{2\pi}\int \frac{dz}{z}\, q(y)\, P(z)\, \frac{2}{\epsilon}$$

(MS-scheme).

- Renaming $q_{phys} \to q$, we finally have

$$F_\Sigma(x, Q^2) = x \int \frac{dz}{z}\, q(y)\left[\delta(1-z) + \frac{\alpha_s}{2\pi}\left(P(z)\ln\frac{Q^2}{\mu^2} + \underset{\substack{\uparrow \\ \text{"coefficient fct."}}}{C(z)}\right)\right]$$

- Observables should not depend on the "factorization scale" μ^2, i.e. $\frac{dF_2}{d\mu^2} = 0$.

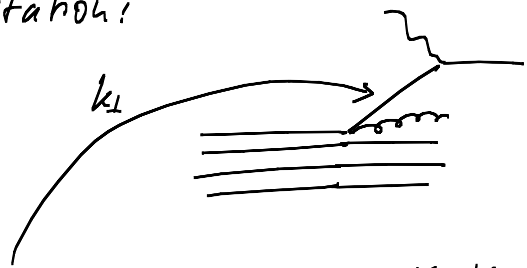
$\Rightarrow \quad q(x) = q(x, \mu^2)$!!

- We find: $$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} P(z)\, q(y,\mu^2) \quad ; \quad y = \frac{x}{z}$$

$$\frac{dq}{d\ln\mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \cdot P \otimes q$$

"convolution".

- Interpretation:

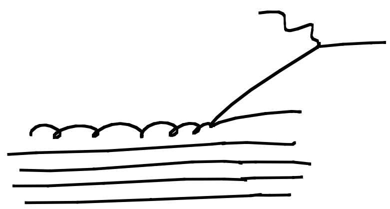


"One of the incoming quarks has "split" into quark (& gluon)."

One can convince oneself that the divergence really comes from $\int_{\mu^2}^{Q^2} \frac{d^2k_\perp}{k_\perp^2}$ of this line.

"factorizes" hard from soft part.

- Analogously, one has a gluon distribution and a splitting fct. for gluons into quarks:

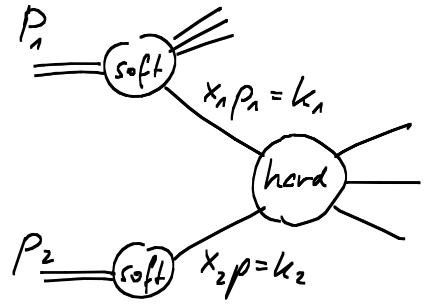


- This gives us the "Altarelli-Parisi" (or "DGLAP") eqs. for the μ^2 dependence of parton distributions. Schematic:

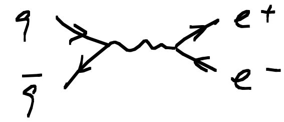
$$\frac{d}{d\ln\mu^2}\begin{pmatrix} q \\ g \end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}$$

(We have discussed just P_{qq} above.)

8.4 Factorization of hard processes in hadronic collisions



In the simplest case, the hard process is $q\bar{q} \to e^+e^-$,



also known as "Drell-Yan process".

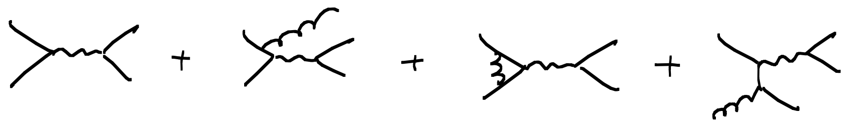
$$\sigma = \sum_{i,j} \int dx_1\, dx_2\, f_i(x_1,\mu^2)\, f_j(x_2,\mu^2)\, \hat{\sigma}_{ij}(k_1, k_2, \alpha_s(\mu^2), Q^2, \mu^2)$$

PDFs — factorization scale — "hard scale" (usually $Q^2 \sim k_1 \cdot k_2$)

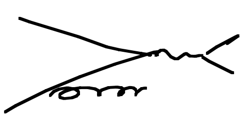
- The crucial claim is that the PDFs appearing above do not depend on which hard process one considers. (In particular, they are the same PDFs appearing in DIS on one hadron.)

- As we have seen, the μ^2-dependence of the PDFs is an $O(\alpha_s)$-effect. Thus, at LO, both the f_i and $\hat{\sigma}$ are μ^2-independent.

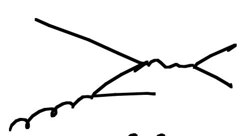
- At NLO, both the f_i and $\hat{\sigma}$ depend on μ^2 in such a way that this dependence cancels out in σ (to order α_s). This structure continues to hold to all orders. Roughly speaking, the IR/collinear divergences in $\hat{\sigma}$ are absorbed in the PDFs, thereby defining their μ^2-dependence.

- For DY:  etc.

- The divergences allow for the calculation of the qq & qg splitting fcts:


qq

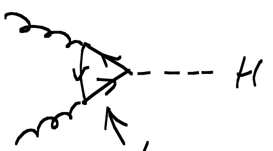
;


qg ("quark in gluon")

- The non-trivial claims made above are known as the "factorization" of the hard cross-section from the soft, hadronic part of the amplitude (the latter being parametrized by the PDFs).

- The crucial technical point behind all of this is the dominance of the log.-divergences
$$\int \frac{dk_\perp^2}{k_\perp^2} ,$$
which arise for both k_1 & k_2 and dominate the calculation. This integral is cut off by μ^2 in the IR and by Q^2 in the UV. In the end, it is convenient to set $\mu^2 \approx Q^2$ to avoid large logs in the result.

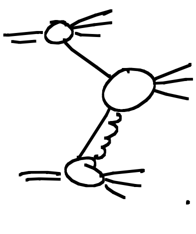
- One of the crucial examples for LHC: Higgs-production by gluon-gluon fusion:



top-quark dominates.

(See also lecture notes of T. Plehn and book by Ellis/Stirling/Webber.)

- Final comment: The above picture holds only at leading order in $1/Q^2$ ("at leading twist"), i.e. corrections of order Λ^2/Q^2 do not in general respect factorization. The corresponding diagrams are, e.g., of the type



.