9 Anomalies

An "Anomaly" is the break-down of a symm. of the class. system after quantization.

9.1 Chiral anomaly – functional integral approach

- Consider a model with a $U(1)$-charged massless Dirac fermion:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi .$$

In addition to the (gauged) $U(1)$-symm. $\psi \to e^{i\alpha}\psi$, this model has the global $U(1)$-symm. $\psi \to \exp(i\alpha\gamma_5)\psi$.

- Let us analyse the trf. properties of the path-integral measure $D\psi D\bar{\psi}$ under a field redefinition $\psi(x) \to \psi'(x) = \int d^4y\, M(x,y)\psi(y)$.

- As we already learned, $D\psi' D\bar{\psi}' = (\det M)^{-1} (\det \bar{M})^{-1} D\psi D\bar{\psi}$,

$$\bar{M} = \gamma^0 M^{+} \gamma^0 .$$

- The specific trfs. $\psi(x) \to e^{i\alpha(x)}\psi(x)$ & $\psi(x) \to e^{i\alpha(x)\gamma^5}\psi(x)$ correspond to $M(x,y) = U(x)\,\delta^4(x-y)$ with $U = \begin{cases} e^{i\alpha} \\ e^{i\alpha\gamma^5} \end{cases}$.

- We $U\bar{U} = U\gamma^0 U^{+}\gamma^0 = \mathbb{1}$ in the first and

$$U\bar{U} = U\gamma^0 U^{+}\gamma^0 = e^{i\alpha\gamma^5}\gamma^0 e^{-i\alpha\gamma^5}\gamma^0 = U^2 \text{ in the second case.}$$

- Hence, for the second, chiral, trf. the measure is not automatically invariant, but rather

$$D\psi D\bar{\psi} \to (\det M)^{-2} D\psi D\bar{\psi} .$$

- Now consider infinitesimal chiral rotations, $\alpha \ll 1$, such that $\quad (M-\mathbb{1})(x,y) = i\alpha(x)\gamma_5\,\delta^4(x-y)$.

- We have $\det M = \exp \ln\det M = \exp \mathrm{tr}\ln M$

$$= \exp \mathrm{tr} \ln(\mathbb{1}+(M-\mathbb{1})) \approx \exp \mathrm{tr}(M-\mathbb{1})$$

$$= \exp i\int d^4x\, \alpha(x)\, \underbrace{\mathrm{tr}\,\gamma_5\,\delta(x-x)}_{\equiv -\frac{1}{2}\mathcal{A}(x)}.$$

"anomaly fct.". $(= 0\cdot\infty$? see below...)

$$\Rightarrow \quad D\psi D\bar\psi \to \exp\left(i\int d^4x\, \alpha(x)\mathcal{A}(x)\right) D\psi D\bar\psi.$$

- Before calculating $\mathcal{A}$ (in a regularized way), let's consider the meaning of a possibly non-zero $\mathcal{A}$:

$$\int D\psi' D\bar\psi' DA_\mu\; O(A_\mu)\, e^{i\int\mathcal{L}(\psi',A_\mu)}$$

$$= \int D\psi D\bar\psi DA_\mu\; O(A_\mu)\, e^{i\int\left(\mathcal{L}(e^{i\alpha\gamma_5}\psi,\,A_\mu) + \alpha(x)\mathcal{A}(x)\right)}$$

- We see that a non-invariance of the measure is equivalent to an (extra) trf. of $\mathcal{L}$ under $\psi \to e^{i\alpha\gamma_5}\psi$:

$\mathcal{L} \to \mathcal{L} + \alpha(x)\mathcal{A}(x) \quad \left[\text{or: } \mathcal{L}(\psi, A_\mu) \;\text{"}\hat{=}\text{"}\; \mathcal{L}(e^{i\alpha\gamma_5}\psi, A_\mu) + \alpha(x)\mathcal{A}(x)\right]$

"under path integral"

- To regularize $\mathcal{A}$, write it as

$$\mathcal{A}(x) = -2\,\mathrm{tr}\left(\gamma^5 f(-\not{D}_x^2/M^2)\right)\delta^4(x-y)\Big|_{y=x}$$

This gives us formally our previous unregularized $\mathcal{A}$ for $M\to\infty$ if $f(0)=1$. (M is just some mass scale, not to be confused with our previous $M(x,y)$)

It also regularizes the δ-fct. if $f(s)\to 0$ as $s\to\infty$

since fast variations, i.e. large values of $\not{D}^2$, are then suppressed. (E.g. $f(s) = e^{-s^2}$ or $1/(1+s)$ ... any smooth fct. will do.)

- Crucial point: We needed to use $\not{D}$ rather than $\not{\partial}$ to avoid breaking the $U(1)$ gauge-invariance.

- Now we find: $\mathcal{A}(x) = -2\int \frac{d^4k}{(2\pi)^4}\, tr\left(\gamma_5 f(-\not{D}_x^2/M^2)\right) e^{-ik(x-y)}\Big|_{y=x}$

Think of $f(-\not{D}_x^2/M^2)$ as Taylor series in $\not{D}_x$ and use

$$\not{D}_x\, e^{ik(x-y)}(\cdots) = e^{ik(x-y)}(i\not{k} + \not{D}_x)(\cdots).$$

$$\Rightarrow \mathcal{A}(x) = -2\int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, tr\left(\gamma_5 f\left(-(i\not{k}+\not{D}_x)^2/M^2\right)\right)\Big|_{y=x}$$

$$= -2\int_k tr\left(\gamma_5 f\left(-(i\not{k}+\not{D}_x)^2/M^2\right)\right)$$

∂_x is non-trivial since it can act on $A_\mu(x)$!

- Let $k \to kM$ and use $-(i\not{k} + \not{D}_x/M)^2 = k^2 - \frac{2ik_\mu D_x^\mu}{M} - \left(\frac{\not{D}_x}{M}\right)^2$.

$$\Rightarrow \mathcal{A}(x) = -2M^4\int_k tr\left(\gamma_5 f\left(k^2 - \frac{2ik\cdot D_x}{M} - \left(\frac{\not{D}_x}{M}\right)^2\right)\right)$$

Observe: – if more than 4 factors of $1/M$ $\to$ get zero for $M\to\infty$.
– if less than 4 powers of $\not{D}$ $\to$ get zero because of trace with γ_5

$$\Rightarrow \mathcal{A}(x) = -\int_k f''(k^2)\, tr\left(\gamma_5 \not{D}_x^4\right)$$

- Wick-rotation $k^0 \to ik^0$: $\int d^4k\, f''(k^2) \to i\int d^4k_E\, f''(-k_E^2) = i\int_0^\infty 2\pi^2 k^3 dk\, f''(-k^2)$

$$= i\pi^2\int_0^\infty s\,ds\, f''(-s) \underset{s=-t}{=} i\pi^2\int_0^{-\infty} t\,dt\, f''(t) = -i\pi^2\int_0^{-\infty} dt\, f'(t) = -i\pi^2 f(t)\Big|_0^{-\infty} = i\pi^2.$$

- $$\not{D}^2 = D_\mu D_\nu \gamma^\mu \gamma^\nu = D_\mu D_\nu \left(\tfrac{1}{2} \{\gamma^\mu, \gamma^\nu\} + \tfrac{1}{2} [\gamma^\mu, \gamma^\nu] \right)$$
$$= D_\mu D^\mu + \tfrac{1}{4} [D_\mu, D_\nu][\gamma^\mu, \gamma^\nu] = D_\mu D^\mu + \tfrac{i}{4} F_{\mu\nu} [\gamma^\mu, \gamma^\nu].$$

$$\Rightarrow \ \mathrm{tr}(\gamma^5 \not{D}^4) = -\frac{1}{16} F_{\mu\nu} F_{\rho\sigma} \, \mathrm{tr}(\gamma^5 \underbrace{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]}_{\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^5 \text{ for } \mu\nu\rho\sigma = 0123})$$

$(\gamma^5)^2 = \mathbb{1}$; $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$

get $2 \cdot 2 \cdot \mathrm{tr}\,\mathbb{1}$ for $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$
& totally antisymmetric

$$\mathrm{tr}(\gamma^5 \not{D}^4) = \frac{1}{16} F_{\mu\nu} F_{\rho\sigma} \, i \cdot 16 \, \varepsilon^{\mu\nu\rho\sigma} \qquad (\varepsilon^{0123} = 1)$$

$$\Rightarrow \ \mathcal{A} = \frac{-i\pi^2}{(2\pi)^4} \cdot i \cdot F_{\mu\nu} F_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} = \underline{\underline{\frac{1}{16\pi^2} F\tilde{F}}} \, .$$

9.2 Anomalous current non-conservation

- Thus, we indeed find a non-vanishing anomaly. This anomaly can be interpreted as an "anomalous" violation of the conservation of the Noether current j_5^μ associated with the global symmetry $\psi \to e^{i\gamma^5\alpha}\psi$:

- The Noether current associated with $\psi \to e^{i\alpha}\psi$ is $j^\mu = \bar{\psi}\gamma^\mu\psi$.
- Analogously, $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. (You can proof the conservation of this current in the usual way, appealing to the Noether theorem.)

- Consider now $\psi \to e^{i\alpha\gamma^5}\psi$ with $\alpha = \alpha(x)$.

$$\delta S = \delta \int d^4x \, \bar{\psi} i \not{\partial} \psi = \int d^4x \, \left(\overline{\psi + i\gamma^5\alpha\psi}\right) i\not{\partial} (\psi + i\gamma^5\alpha\psi)$$
$$- \int d^4x \, \bar{\psi} i \not{\partial} \psi$$
$$= \int d^4x \, \bar{\psi} \, i\gamma^\mu (\partial_\mu \alpha) i\gamma^5 \psi = -\int d^4x \, j_5^\mu \, \partial_\mu \alpha$$

$$= \int d^4x \, (\partial_\mu j_5^\mu) \cdot \alpha$$

(Note: This does in fact proof current conservation since the action is stationary, including the variation $\delta\psi = i\gamma^5\alpha\psi$, if the field satisfy EOMs)

- Given our previous calculation, we find

$$0 = \delta \int D\psi D\bar\psi \, e^{iS} = \int D\psi D\bar\psi \, e^{iS + i\int d^4x \, (\alpha\mathcal{A} + (\partial_\mu j_5^\mu)\alpha)} - \int D\psi D\bar\psi \, e^{iS}$$

subst. of variables
$\psi \to e^{i\gamma^5\alpha}\psi$

$$\Rightarrow \quad \partial_\mu j_5^\mu = -\frac{1}{16\pi^2} \, \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

(under the path integral, i.e., for the QFT operators!)

- This anomalous non-conservation of the axial-vector current ($\bar\psi\gamma^\mu\gamma^5\psi$ is an axial vector) generalizes in several ways.

(1) We can write $\psi = \psi_L + \psi_R = \frac{1-\gamma_5}{2}\psi + \frac{1+\gamma_5}{2}\psi$.

Instead of ψ_R, we can use the l.h. Dirac fermion

$$\underbrace{(\psi_R)^c}_{\text{charge conjugation}} = \begin{pmatrix} \chi_w \\ 0 \end{pmatrix}$$

$\left[\psi = \begin{pmatrix} \psi_w \\ \bar\chi_w \end{pmatrix};\ \psi_L = \begin{pmatrix} \psi_w \\ 0 \end{pmatrix};\ \psi_R = \begin{pmatrix} 0 \\ \bar\chi_w \end{pmatrix};\ \psi^c = \begin{pmatrix} \chi_w \\ \bar\psi_w \end{pmatrix};\ w \equiv \text{"weyl"}\right]$

as our fundamental field.

Thus, $\bar\psi i\not{D}\psi = \bar\psi_L i\not{D}\psi_L + \overline{\psi_R^c}\, i\not{D}\psi_R^c$.

	ψ_L	ψ_R^c
U(1)–V charge	+1	−1
U(1)–A charge	−1	−1 .

; $j_5^\mu = j_L^\mu + j_{R^c}^\mu$

(obvious)

Furthermore, $\mathcal{A} = \mathcal{A}_L + \mathcal{A}_{R^c} = 2\mathcal{A}_L$ (Since ψ_L & ψ_R^c are identical, up to the U(1)–V charge. But this charge enters $\mathcal{A}$ quadratically)

$$\Rightarrow \partial_\mu j_5^\mu = \partial_\mu j_L^\mu + \partial_\mu j_{R^c}^\mu = -\frac{1}{16\pi^2} F\tilde{F} = -\frac{1}{32\pi^2} F\tilde{F}\Big|_{\text{from } \psi_L} - \frac{1}{32\pi^2} F\tilde{F}\Big|_{\text{from } \psi_R}$$

$$\Rightarrow \boxed{\partial_\mu j^\mu = -\frac{1}{32\pi^2} F\tilde{F}}$$ In a theory with just one l.h. fermion.

② Now, working with l.h. fermions only, we can replace

$$\psi_L \to \psi_{Li} \,,$$

with i running over the basis of some repres. of a non-abelian symm. group (e.g. $SU(3)\times SU(2)\times U(1)$ of the SM). There can be many "simple" factors like $SU(N)$ and many $U(1)$s. We can collect all generators of this large symmetry an call them T_a (Think of $T_a = \frac{\lambda_a}{2}$ of $SU(3)$ or $T_a = \mathbb{1}$ for a $U(1)$: $\psi_{Li} \to e^{i\alpha}\psi_{Li}$, in full generality.)

- Our anomaly calculation generalizes easily, by $tr(\cdots)$ splitting into a product $tr(--) = tr_{Dirac}(--)\cdot tr_{group}(--)$.
- The result for the anomaly of the current associated with T_a ($\psi_{Li} \to (e^{iT_a})_{ij}\psi_{Lj}$; no γ^5 needed!) reads:

$$\partial_\mu j_a^\mu = -\frac{1}{32\pi^2} D_{abc}\, \varepsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma}$$

$$\text{where} \quad \underbrace{D_{abc}}_{\text{totally symmetric}} = \frac{1}{2} tr\left[T_a \{T_b, T_c\}\right]$$

totally symmetric (easy to check using $tr(AB) = tr(BA)$)

anticommutator justified by contraction with $\varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma}$

cf. Alvarez-Gaume, Ginsparg, Nucl. Phys. B243 (1984) 449.

- This can be direcly applied to the SM, where we have:

	$SU(3)$	$SU(2)$	$U(1)$
$\binom{u}{d}_L$	3	2	$-1/6$
u_R^c	$\bar{3}$	1	$2/3$
d_R^c	$\bar{3}$	1	$-1/3$
$\binom{\nu_e}{e}_L$	1	2	$1/2$
e_R	1	1	-1

- Where we can prove by simple algebra that all anomalies associated with the gauge-group-generators vanish (for each generation seperately).

- <u>Crucial:</u> We can <u>not</u> tolerate an anomaly for a gauged symm. since renormalizability would be lost. (We need the gauge symm. to bring the propagator to the form $\eta^{\mu\nu}/p^2$, such that divergences are not worse than for a scalar.)

9.3 Chiral anomaly – Feynman-diagram-approach – 2d-example.

Consider QED in $d=2$: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \not{D} \psi$;

$$\mu,\nu = 0,1 \;;\; \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;;\; \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;;\; \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

(2-component spinor)

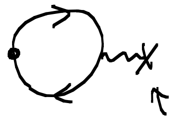
The V & A currents are $\underbrace{\bar{\psi}\gamma^\mu\psi}_{j^\mu}$ & $\underbrace{\bar{\psi}\gamma^\mu\gamma^5\psi}_{j_5^\mu}$, as in $d=4$.

- The crucial simplification arises from the identity

$$\gamma^\mu \gamma^5 = -\varepsilon^{\mu\nu}\gamma_\nu \qquad (\varepsilon^{01} = 1).$$

This implies $j_5^\mu = -\varepsilon^{\mu\nu} j_\nu$.

- We want to calculate $\langle j^\mu \rangle$ (from which $\langle j_5^\mu \rangle$ follows, see above) in the presence of a background field A_μ at leading order in e:



A_μ (no propagator, use directly $\bar\psi \, ie \slashed{A} \psi$.)

- This gives $\langle j^\mu(q) \rangle = (-ie)^{-1}\, i\Pi^{\mu\nu}(q)\, A_\nu(q)$. Fourier trf. of $A_\nu(x)$

Fourier trf. of $\langle j^\mu(x) \rangle_{A(q')}$ with $\delta^4(q-q')$ extracted.

- We already did the calculation of $\Pi_{\mu\nu}$ in d dimensions:

$$\Pi^{\mu\nu}(q) = -4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \cdot \frac{-\frac{2}{d}\eta^{\mu\nu}k^2 + \eta^{\mu\nu}k^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}x(1-x)q^2}{(k^2+\Delta)^2}$$

euclidean

$\Delta = -x(1-x)q^2$

this comes from

$k^\mu k^\nu \to \frac{1}{d} k^2 \eta^{\mu\nu}$

$$= -e^2 \int_0^1 dx \, \frac{\mathrm{tr}(\mathbb{1})}{(4\pi)^{d/2}} \cdot \frac{1}{\Delta^{2-d/2}} \cdot \Big\{ \left(1-\tfrac{d}{2}\right)\Gamma\left(1-\tfrac{d}{2}\right)\left(-\Delta\eta^{\mu\nu}\right) +$$

$$+ \Gamma\left(2-\tfrac{d}{2}\right)\left[-\Delta\eta^{\mu\nu} - 2x(1-x)q^\mu q^\nu\right]\Big\}$$

This part comes from the k^2 in numerator, which gives a quadratic divergence in $d=4$, i.e. a pole in $d=2$.

This pole is cancelled by prefactor $1-\frac{d}{2}$: $\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) = \Gamma\left(2-\frac{d}{2}\right)$.

- Together, we find

$$= -2\,\mathrm{tr}(\mathbb{1})\, e^2 \int_0^1 dx \, \frac{x(1-x)}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \cdot (q^2 \eta^{\mu\nu} - q^\mu q^\nu)$$

$$= (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \frac{2e^2}{4\pi} \cdot \underset{\substack{\uparrow \\ \text{from } \mathrm{tr}(\mathbb{1}) = 2}}{2} \cdot \frac{1}{q^2} \qquad \text{using} \int_0^1 dx \, \frac{x(1-x)}{-x(1-x)q^2} = -\frac{1}{q^2}$$

$$= \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{e^2}{\pi}$$

(Note: This does not vanish at $q^2 \to 0$ since Π from $\Pi_{\mu\nu} = (\eta_{\mu\nu} q^2 - q_\mu q_\nu)\Pi$ has a pole $\sim \frac{1}{q^2}$.
$\Rightarrow$ Photon gets mass in $d=2$. This is interesting, but we will not pursue this issue.)

- Our central result is

$$\langle j_5^\mu \rangle = -\varepsilon^{\mu\nu} \langle j_\nu \rangle = -\varepsilon^{\mu\nu} (-ie)^{-1} i \Pi_{\nu\rho} A^\rho$$

$$= -\varepsilon^{\mu\nu} (-ie)^{-1} i \left(\eta_{\nu\rho} - \frac{q_\nu q_\rho}{q^2}\right) \frac{e^2}{\pi} A^\rho$$

$$= \varepsilon^{\mu\nu} \frac{e}{\pi} \left(A_\nu - \frac{q_\nu q^\rho}{q^2} A_\rho\right).$$

- Thus, while obviously $q_\mu \langle j^\mu \rangle = 0$, we have for the axial current

$$q_\mu \langle j_5^\mu \rangle = \frac{e}{\pi} \varepsilon^{\mu\nu} q_\mu A_\nu \neq 0 .$$

- The Fourier transform is

$$\underline{\underline{\partial_\mu j_5^\mu = \frac{e}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}}}$$

(Note the structural similarity to our 4d-path-int. result!)

Note: Anomalies arise in all even dimensions. (We have an indep. γ^{d+1}-matrix only for even d – for d odd there is no chirality.
Example: $d=5$; γ-matrices: $\gamma^0, \dots, \gamma^3, \gamma^5 = i\gamma^0 \cdots \gamma^3$; $\{\gamma^M, \gamma^N\} = 2\eta^{MN}$.
To define chirality, we would need "γ^6" $\sim \gamma^0 \cdots \gamma^3 \gamma^5$. But this is $\sim \mathbb{1}$.
$\Rightarrow$ No chirality; no chiral anomaly.) The structure is always

$$\sim \varepsilon^{\mu_1\mu_2 \cdots \mu_{d-1}\mu_d} F_{\mu_1\mu_2} \cdots F_{\mu_{d-1}\mu_d}$$

(abelian case).

- Before leaving the 2d-case, let us briefly discuss the technical origin of the anomaly in the Feynman-diagram approach:

Class. $U(1)$-symm. (chiral or not) $\Rightarrow$ $\partial_\mu j^\mu_{(5)} = 0$.
This survives all formal manipulations under the path-integral and hence leads to corresponding Ward identities for Green's fcts. $\Rightarrow$ $q_\mu \langle j^\mu_{(5)}(q) \rangle = 0$ even if we now include quantum corrections in the form of Feynman diagrams.

Problem: To show "$\dots = 0$" diagramatically, we need to formally manipulate the expressions (change order of int.s; shift int. variables etc.). This is only justified as long as everything is convergent. Once we encounter divergent expressions this can go wrong (and it does!). Indeed, this is not unexpected since we encountered the anomaly from the perspective of the measure $D\psi D\bar\psi$, especially its UV-regularization. But we only become sensitive to that region ($k \to \infty$) once we try to really evaluate a divergent diagram.

- We can make this more explicit recalling our calculation of $\Pi_{\mu\nu}$:

① $\int d^2k \, \frac{k^2}{(k^2+\Delta)^2} \eta^{\mu\nu}$ ② $\int d^2k \, \frac{q^\mu q^\nu}{(k^2+\Delta)^2}$

prefactor

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\underbrace{(1-\tfrac{d}{2})\,\Gamma(1-\tfrac{d}{2})}\,\eta^{\mu\nu} \qquad\qquad \underbrace{\Gamma(2-\tfrac{d}{2})}\, q^\mu q^\nu$$

Divergence cancelled by prefactor; finite but regularization dependent — finite — $(d=2)$

$$\Pi^{\mu\nu} \sim \left(A\eta^{\mu\nu} - B\, q^\mu q^\nu / q^2\right).$$

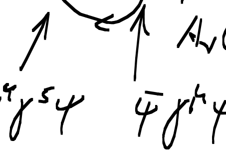
"Vector"-gauge-inv. demands $A = B$. Then $\partial_\mu j^\mu = 0$ & $\partial_\mu j_5^\mu \neq 0$. (This is automatically realized by dim. reg. since it respects "Vector" gauge invariance. We will see later more explicitly why it has trouble with "Chiral" gauge invariance.)

- Alternatively, we could have enforced $\partial_\mu j_5^\mu = 0$ by appropriate regularization of A $(A = 0)$. But then $\partial_\mu j^\mu \neq 0$ and our theory is plainly inconsistent if j^μ is gauged.

9.4 Chiral anomaly – Feynman-diagram approach – $d=4$

- Note that no equ. analogous to $j_5^\mu = -\varepsilon^{\mu\nu} j_\nu$ $(d=2)$ exists in $d=4$.
- Nevertheless, we can perform a calculation analogous to Sect. 9.3:

$$\langle j_5^\mu(q)\rangle = \mathcal{M}^{\mu\nu}(q)\, A_\nu(q) \quad \longleftrightarrow$$



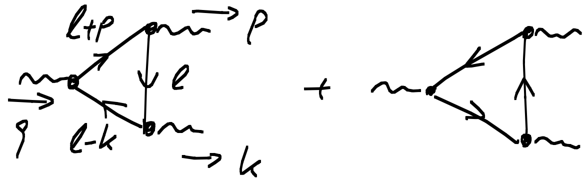
$A_\nu(q)$

$\bar\psi\gamma^\mu\gamma^5\psi$ $\bar\psi\gamma^\mu\psi$

(with δ-fct. extracted)

- It will become clear below, that we won't find the anomaly. To find it, we need to go to next order:

$$\langle p,k|j_5^{\mu}(q)|0\rangle = (2\pi)^4\delta^4(\ldots)\mathcal{M}^{\mu\nu\rho}(p,k)\,\varepsilon_\nu^*(p)\,\varepsilon_\rho^*(k)$$

$\longleftrightarrow$ 

(This is just the common notation. The ext. lines are amputated. The "q"- ext. line really doesn't exist if j_5^μ is not gauged.)

- We will discuss the calculation very briefly, focussing only on the crucial conceptual points:

1st. diagram mult. with q_μ $(\sim \partial_\mu j_5^\mu)$ $\quad \sim q_\mu \int_\ell \mathrm{tr}\left[\gamma^\mu\gamma^5 \frac{1}{\not{\ell}-\not{k}}\gamma^\rho\frac{1}{\not{\ell}}\gamma^\nu\frac{1}{\not{\ell}+\not{p}}\right]$

Use $\not{q}\gamma^5 = (\not{\ell}+\not{p}-\not{\ell}+\not{k})\gamma^5 = (\not{\ell}+\not{p})\gamma^5+\gamma^5(\not{\ell}-\not{k})$

$$\Rightarrow \qquad = \int_\ell \mathrm{tr}\left[\gamma^5\frac{1}{\not{\ell}-\not{k}}\gamma^\rho\frac{1}{\not{\ell}}\gamma^\nu + \gamma^5\gamma^\rho\frac{1}{\not{\ell}}\gamma^\nu\frac{1}{\not{\ell}+\not{p}}\right]$$

Use $[\gamma^5\gamma^\rho = -\gamma^\rho\gamma^5]$ & $[\ell \to \ell+k$ in 1st term only$]$

$$\Rightarrow \qquad = \int_\ell \mathrm{tr}\left[\gamma^5\frac{1}{\not{\ell}}\gamma^\rho\frac{1}{\not{\ell}+\not{k}}\gamma^\nu - \gamma^5\frac{1}{\not{\ell}}\gamma^\nu\frac{1}{\not{\ell}+\not{p}}\gamma^\rho\right]$$

This is antisymm. under $p,\nu \leftrightarrow k,\rho$. Since the 2nd diagram follows from the 1st by $p,\nu \leftrightarrow k,\rho$, their sum is <u>zero</u>. But this result of formal, unregularized manipulations is <u>wrong</u>!

(Shifting int. variable in divergent integral is the specific problem.)

Now: dim.-reg. . Problem: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. What does this mean in d dims.??

- 't Hooft / Veltman prescription: use "formal" γ^5 with

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{for } \mu = 0 \cdots 3$$

$$[\gamma^5, \gamma^\mu] = 0 \quad \text{for } \mu = 5 \cdots d$$

(in d dims., it is common to use $\mu, \nu, \rho, \cdots = 0, 1, 2, 3, 5, 6, \cdots, d$)

- Write $l = \underbrace{l_\parallel}_{\substack{\text{in dims.}\\ 0\cdots 3}} + \underbrace{l_\perp}_{\substack{\text{in other}\\ \text{dims.}}}$

Now: $\not{q}\gamma^5 = (\not{l} + \not{p} - \not{l} + \not{k})\gamma^5 = (\not{l} + \not{p} - \not{l}_\parallel - \not{l}_\perp + \not{k})\gamma^5$

$$= (\not{l} + \not{p})\gamma^5 + \gamma^5(\not{l}_\parallel - \not{l}_\perp - \not{k})$$

$$= \underbrace{(\not{l} + \not{p})\gamma^5 + \gamma^5(\not{l} - \not{k})}_{\text{will give zero as before}} - \underbrace{2\gamma^5 \not{l}_\perp}_{\substack{\text{new contribution!}\\ \text{(same from both diagrams)}}}$$

$$\Rightarrow \quad q^\mu \cdot (\text{1st + 2nd diagr.}) \sim \int_l \mathrm{tr}\left(\gamma^5 \not{l}_\perp \frac{1}{\not{l} - \not{k}} \gamma^\rho \frac{1}{\not{l}} \gamma^\nu \frac{1}{\not{l} + \not{p}}\right)$$

- Write $\frac{1}{\not{l} - \not{k}} = \frac{\not{l} - \not{k}}{(l-k)^2}$ etc.

- Introduce Feynman parameters using the more general formula

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + \cdots + x_n A_n]^n}$$

- focus on $\mathrm{tr}(\gamma^5 \not{\ell}_\perp (\not{\ell}-\not{k})\gamma^\rho \not{\ell}\gamma^\nu(\not{\ell}+\not{p}))$; to get non-zero result, need to keep $\not{k}, \not{p}, \gamma^\rho, \gamma^\nu$ (because of γ^5) and $\not{\ell}_\perp$-part of some $\not{\ell}$ (because of explicit $\not{\ell}_\perp$-factor).

- $$\int_\ell \frac{\not{\ell}_\perp \not{\ell}_\perp}{(\ell^2-\Delta)^3} = \frac{d-4}{d} \underbrace{\int_\ell \frac{\ell^2}{(\ell^2-\Delta)^3}}_{\sim \Gamma(2-\frac{d}{2}) \sim \frac{1}{d-4}} \overset{=\!=}{\underset{d=4}{}} \frac{-i}{2(4\pi)^2}$$

pole cancelled by prefactor

$$\Rightarrow \text{ in total: } \quad q_\mu \cdot (\text{1st} + \text{2nd diagr.}) = \frac{2ie^2}{(4\pi)^2} \mathrm{tr}[\gamma^5 \not{k}\gamma^\rho \not{p}\gamma^\nu]$$

$$= \frac{e^2}{2\pi^2} \varepsilon^{\alpha\rho\beta\nu} k_\alpha p_\beta .$$

$$\Rightarrow \langle p, k | \partial_\mu j^{\mu 5}(0) | 0 \rangle = -\frac{e^2}{2\pi^2} \varepsilon^{\alpha\rho\beta\nu} (-ik_\alpha)\varepsilon^*_\nu(p)(-ip_\beta)\varepsilon^*_\rho(k)$$

by undoing Fourier-trf. from x to q and setting $x=0$.

$\partial_\alpha A_\rho$ etc.

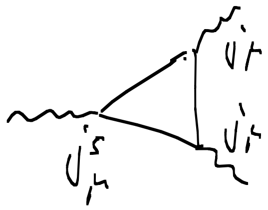
$$= -\frac{e^2}{16\pi^2} \langle p, k | \varepsilon^{\alpha\nu\beta\rho} F_{\alpha\nu} F_{\beta\rho}(0) | 0 \rangle$$

(e^2 appears since here $D_\mu = \partial_\mu + ieA_\mu$, as opposed to $D_\mu = \partial_\mu + iA_\mu$ in path-int. derivation above)

In agreement with path-int. derivation (Note: Anomalies are "saturated" at 1-loop).

Final comments:

① The "triangle diagram"



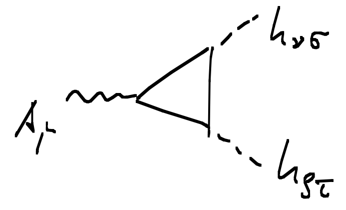
makes our original discussion of the chiral $U(1)$ anomaly particularly intuitive: It is this diagram that violates the chiral-current-conservation (in presence of a non-chiral gauged $U(1)$). This is the famous "Adler-Bell-Jackiw" (or ABJ) anomaly.

② Replacing the vertices by (any of the) generators of an (in general non-abelian) gauge symm., we recover intuitively the condition

$$\mathrm{tr}\left(T_a\{T_b, T_c\}\right) = 0$$

discussed before. (In particular, for a $U(1)$ and l.h. fermions labelled by i, we must have $\sum_i q_i^3 = 0$.)

③ This also extends to the coupling of chiral fermions to gravity, where the anomaly associated with the diagram



$$(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu})$$

gives rise to the extra condition $\sum_i q_i = 0$. (Which is also satisfied in the SM).

④ We already noted that the QCD lagrangian (with zero quark masses) has a global chiral $U(1)$ symm. with current

$$j^\mu_5 = \bar{Q}\gamma^\mu\gamma^5 Q. \qquad Q = \binom{u}{d}$$

Analogously, it has a global $SU(2)$ symm. (isospin) with currents

$$j^{\mu a}_5 = \bar{Q}\gamma^\mu\gamma^5\tau^a Q.$$

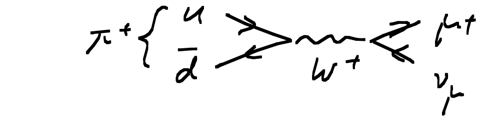
This current has non-trivial overlap with the pions:

$$\langle 0|j_5^{\mu a}(x)|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ipx}$$

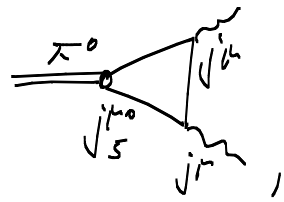
$\pi^{0,\pm}$

pion decay constant; can be determined e.g. from

$\pi^+ \to \mu^+ \nu_\mu$



Using the anomaly of $j_5^{\mu 0}$, which is introduced through the diagram



we can determine the decay rate $\pi^0 \to \gamma\gamma$, which thus provides a (successful!) experimental test of the anomaly calculations.

③ A completely different (but also very important) anomaly is the "scale-invariance" or "trace" anomaly:

The underlying symm. is the change of all length scales of some exp. setup by some universal constant factor. (Equivalently, we could change the definition of the meter (measuring everything in meters) and demand that no observable will change.

A theory which classically has this symm. is QCD,

$$\mathcal{L} = -\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu},$$

which is obvious since $\mathcal{L}$ contains no dimensionful constant.
(E.g. a quark mass term would break this symm. explicitly.)

- However, we know that $\frac{dg}{d\ln\mu^2} = \beta(g) \neq 0$, hence

$$\mathcal{L} = -\frac{1}{2g^2(\mu)} \operatorname{tr} F_{\mu\nu}F^{\mu\nu},$$

i.e. scale invariance is broken by quantum corrections (anomaly!). The name "trace anomaly" comes from the fact that the energy-momentum tensor

$$T^{\mu\nu} \sim \frac{\delta}{\delta g_{\mu\nu}} S,$$

and in particular $T^{\mu}{}_{\mu}$ ("trace") measures the non-invariance under rescalings $g_{\mu\nu} \to c\, g_{\mu\nu}$ $\left(\langle T^{\mu}{}_{\mu} \rangle \neq 0 \right)$.