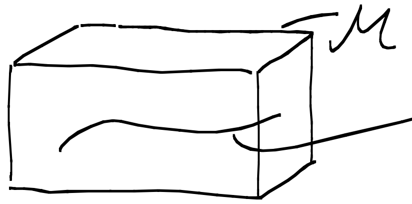
# 2. The classical bosonic string

## 2.1 Relativistic point particle




worldline $\gamma$ $\leftarrow X^\mu(\tau)$ ; $\mu = 0 \dots D-1$
(parameterized by $\tau$)

$$S_{NG} = -m \int_\gamma ds \quad ; \quad ds^2 = -\eta_{\mu\nu}\, dX^\mu dX^\nu \quad ; \quad \hbar = c = 1$$

(NG stands for "Nambu-Goto", in analogy to the corresponding form of the string action (see below))

- $ds$ is the invariant length in $\mathcal{M}$
- with $dX^\mu = \frac{dX^\mu}{d\tau} d\tau = \dot{X}^\mu d\tau$ we have

$$S_{NG} = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$$

- $S_{NG}$ is invariant under reparameterizations $\tau \longrightarrow \tau'(\tau)$ (arbitrary function) $\rightarrow$ problems
- with the special parameter choice $\tau = s$ ($s$ being the proper time of the particle, cf. $ds$ above), the equation of motion (EOM) is $\ddot{X}^\mu = 0$.

$\rightarrow$ problems

- the non-relativistic limit is $S_{NG} \simeq \int d\tau \left(\frac{m}{2}\vec{v}^2 - m\right)$

An equivalent formulation without "$\sqrt{\;}$" is provided by the "Polyakov action" (again, the name really belongs to the corresponding string action):

$$S_P = \frac{m}{2}\int d\tau \sqrt{h_{\tau\tau}} \left(h_{\tau\tau}^{-1} \frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} - 1\right),$$

where $h_{\tau\tau} = h_{\tau\tau}(\tau)$ is the "world line metric".

<u>Aside:</u> let $g_{ab}$ be the metric of a manifold parameterized by $y^a$; $a = 1 \ldots n$. Then $\int d^n x \sqrt{-\det g}$ is reparameterization invariant. Also, with $g^{ab} \equiv (g^{-1})_{ab}$, $g^{ab} \frac{\partial X^\mu}{\partial y^a}\frac{\partial X_\mu}{\partial y^b}$ is invariant.

The above is just the special case of $n = 1$.

To show equivalence to $S_{NG}$, "integrate out" $h_{\tau\tau}$:

- EOM for $h_{\tau\tau} \Rightarrow h_{\tau\tau} = -\dot{X}^\mu \dot{X}_\mu$
- plugging this into the original action, one finds

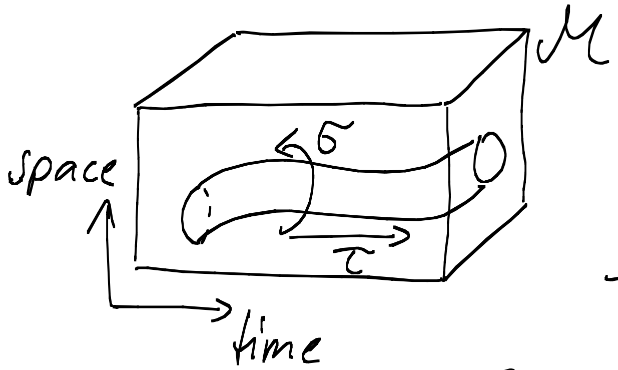
$$S_P[X^\mu, h_{\tau\tau} = -\dot{X}^2] = S_{NG}[X^\mu]$$

$\rightarrow$ problems

We could go on to quantize this. The resulting quantized states can be identified with the 1-particle states in the Fock space of scalar-field QFT

($\to$ Zwiebach's book)

## 2.2 The bosonic string action




The WS $\Sigma$ is described by functions $X^\mu(\tau,\sigma)$; $\mu = 0 \dots D-1$.

In analogy to the point-particle:

$$S_{NG} = -T \int_\Sigma df$$

$\leftarrow$ area of $\Sigma$ as submanifold of $\mathcal{M}$

string tension ($[T] = [mass^2]$)

Also in analogy to point particle:

let $(\tau,\sigma) = (\xi^1, \xi^2) = \xi$ ; introduce WS metric $h$:

$$ds^2 = h_{ab}\, \xi^a \xi^b$$

Polyakov action:

$$S_p = -\frac{T}{2}\int_\Sigma d^2\xi \sqrt{-\det h}\; h^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

Comments: • the overall "$-$" (relative to the point particle case) is linked to our choic $h_{\tau\tau} > 0$

before and $h_{\tau\tau} < 0$ (in analogy to $\eta_{\mu\nu}$) here.

- in the point particle case we had an extra term, the analog of which would be

$$\int_\Sigma d^2\xi \sqrt{-\det h} \, ,$$

corresponding to a WS cosmological constant. Here, it would spoil an important symmetry ($\to$ below).

Equivalence to $S_{NG}$:

The EOM for $h$ is derived from the requirement $\delta \mathcal{L}_P = 0$ under variation of $h$, i.e.

$$0 \stackrel{!}{=} \delta\left\{\sqrt{-\det h}\, h^{ab} G_{ab}\right\} \quad \text{with } G_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$$

$$\Rightarrow 0 = -\frac{\delta \det h}{2\sqrt{-\det h}} h^{ab} G_{ab} + \sqrt{-\det h}\, \delta h^{ab} G_{ab}$$

Aside: For any matrix $A$ with $\det A \neq 0$ we have

$$\delta \det A = (\det A)\, \mathrm{tr}(A^{-1}\delta A) \qquad \to \text{problems}$$

$$\delta \det h = (\det h)\, h^{cd}\, \delta h_{cd} = -(\det h)\, h_{cd}\, \delta h^{cd}$$

$$\Rightarrow 0 = -\frac{1}{2}\sqrt{-\det h}\, h_{ab}\, \delta h^{ab}\, h^{cd} G_{cd} + \sqrt{-\det h}\, \delta h^{ab} G_{ab}$$

$$\Rightarrow \quad \frac{1}{2} h_{ab} h^{cd} G_{cd} = G_{ab}$$

take determinant: $\frac{1}{4} (h^{cd} G_{cd})^2 \det h = \det G$ ;

$$h^{cd} G_{cd} = \frac{2\sqrt{-\det G}}{\sqrt{-\det h}} \quad \Rightarrow \mathcal{L}_p = -T \int_\Sigma d^2\xi \sqrt{-\det G} .$$

Now, since $G_{ab}$ is the metric on the WS <u>induced from $\mathcal{M}$</u> ($\to$ problems), this is indeed equal to

$-T \int_\Sigma d\bar{f}$ , and the equivalence is proven.

<u>Comment:</u> $S_p = -\frac{T}{2} \int d^2\xi \sqrt{-\det h}\, h^{ab} \partial_a X^\mu \partial_b X_\mu$ is a diffeomorphism invariant action for $D$ free scalar fields in 2 dimensions. It would be natural to add a term $\sim \mathcal{R}_2$ (2-dim. Ricci scalar or "Einstein-Hilbert term) to promote this to 2-dim. general relativity. Such a term is a total derivative in 2 dim.s (i.e. it does not affect the EOM for $h$). However, it is sensitive to the global topology of the WS. It "counts the holes" so that its coefficient is related to $g_s$ (see the above discussion of perturbation theory).

## 2.3 Symmetries and EOMs

1) as already mentioned, $S_p$ is invariant under diffeomorphisms: $\xi^a \to \xi'^a(\xi)$ $\to$ problems

2) D-dimensional Poincare invariance can be viewed as an internal symm. of the 2d-field theory:

$$X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu + V^\mu \quad ; \quad \Lambda \in SO(1,D-1).$$

3) Weyl invariance:

$$h_{ab}(\xi) \longrightarrow \underset{\substack{\uparrow \\ \text{arbitrary scalar fct. on WS}}}{\varphi(\xi)}\, h_{ab}(\xi) \equiv e^{2\omega(\xi)} h_{ab}(\xi)$$

(This is special to 2 dims.; it would not hold for objects with more dims., such as "branes"; it would also be spoiled by the cosmol. constant term mentioned earlier.)

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As in GR, an important object is the energy-momentum-tensor:

- in GR: $T^{MN} = \dfrac{1}{\sqrt{-g}} \dfrac{\delta S}{\delta g_{MN}}$
- here: $T^{ab} = \dfrac{4\pi}{\sqrt{-h}} \dfrac{\delta S_p}{\delta h_{ab}} = \dfrac{4\pi}{\sqrt{-h}} \dfrac{\partial \mathcal{L}_p}{\partial h_{ab}}$

($h \equiv \det h$; the "$4\pi$" is specific convention used in string theory)

$$T^{ab} = 2\pi T \left( G^{ab} - \frac{1}{2} h^{ab} h^{cd} G_{cd} \right) \qquad \rightarrow \text{problems}$$

(the calculation is similar to our derivation of the EOM for $h$ above)

<u>Note</u>: $T^a{}_a \equiv 0$ (without use of EOM)

<u>problem</u>: derive this from the symmetries of the action.

As in GR, diff.-invariance implies $\underset{\substack{\uparrow \\ \text{covar. derivative}}}{D_a} T^{ab} = 0$.

<u>EOMs</u>: $h$: $T^{ab} = 0$

$X$: $\Box X^\mu = 0$ $\quad (D^a \partial_a X^\mu = 0)$

## <u>2.4 Gauge choice</u>

(by gauge we here refer to the diff.- and Weyl-inv.)

<u>flat</u> or <u>unit</u> gauge: $h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Why is such a restrictive choice always possible (at least locally)?

1) <u>rough argument</u>: $h$ has 3 functional degrees of freedom

- diff.-invariance: $\xi^a \to \xi^a + \epsilon^a$ : 2 d.o.f.
- Weyl-invariance: $h_{ab} \to e^{2\omega} h_{ab}$ : 1 d.o.f.

$\Rightarrow$ enough freedom to bring $h$ to an arbitrary fixed form.

2) slightly better argument:

$$\sqrt{-h'}\,R' = \sqrt{h}\,(R - 2D^2\omega) \quad \text{where } h'_{ab} = e^{2\omega} h_{ab}$$

an $R = \mathcal{R}_2$ is the 2-dim. Ricci scalar.

- find $\omega$ such that $2D^2\omega = R$ (This is always possible since $\omega$ can be seen as the electrostatic potential for a charge distribution $R$.)

$$\Rightarrow R' = 0 \Rightarrow R_{abcd} = \tfrac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})R = 0$$

$\to$ problems

$\Rightarrow$ flat 2-dim. space $\Rightarrow$ can always chose parameterization such that

$$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the flat gauge, the X-EOMs are very simple:

$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0$ (Klein-Gordon-eqs. in 2 dims.)

furthermore: let $\sigma^\pm = \tau \pm \sigma$ ; $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

( <u>check:</u> $\partial_+ \sigma^\pm = \frac{1}{2}(1+(\pm 1)) = 1/0$

$\partial_- \sigma^\pm = \frac{1}{2}(1-(\pm 1)) = 0/1$ ✓ )

$$ds^2 = d\tau^2 - d\sigma^2 = -d\sigma^+ d\sigma^-$$

$$h_{++} = h_{--} = 0 \ ; \ h_{+-} = -\tfrac{1}{2} \ ; \ h^{+-} = -2$$

$$\Box = 2h^{+-}\partial_+\partial_- = -4\partial_+\partial_-$$

$\Rightarrow$ EOMs simplify further to $\partial_+\partial_- X^\mu = 0$

$\Rightarrow$ general solution: $X^\mu = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$

For the <u>closed string</u> we also have $X^\mu(\tau,\sigma) = X^\mu(\tau, \sigma+\pi)$

These are the conventions of GSW.
($\pi \to 2\pi$ is probably more wide-spread)

In terms of $X_{L,R}$ this means

$$X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) = X_L^\mu(\sigma^+ + \pi) + X_R^\mu(\sigma^- - \pi)$$

- obviously, this is fulfilled if $X_{L,R}$ are periodic with period $\pi$.
- however, the most general possibility includes a shift by a constant: $X_L^\mu(\sigma^+) = X_L^\mu(\sigma^+ + \pi) + c$

$$X_R^\mu(\sigma^-) = X_R^\mu(\sigma^- - \pi) - c$$

$\Rightarrow X_{L/R}$ can be periodic functions + an extra term linear in $\sigma^{+/-}$:

<u>general solution:</u>

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{\ell^2}{2}p^\mu\sigma^+ + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-2in\sigma^+}$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{\ell^2}{2}p^\mu\sigma^- + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-2in\sigma^-}.$$

Here $\ell = \sqrt{2\alpha'} = \frac{1}{\sqrt{\pi T}}$ is the "string length".

$\alpha'$ is known as the Regge slope, which goes back to the days of ST as a model of strong interactions (see later).

<u>Reality implies:</u> $x^\mu, p^\mu$ real ; $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ (same with "~")

$X_{L/R}$ can be visualized as left/right-moving waves on the circle ($S^1$). For the sum we have

$$X^\mu = \underbrace{x^\mu + \ell^2 p^\mu \tau + \cdots}_{\text{linear motion in target space + fluctuations}}$$

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A more general gauge is the <u>conformal gauge</u>, where $h_{ab} = e^{2\omega(\xi)}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. It can be achieved using the diffeomorphisms alone.