4 Light cone quantization

4.1 Light cone gauge

- flat gauge: $h_{ab} = \eta_{ab}$ (2-dim. Minkowski metric)
- residual freedom: There exist certain combinations of diffeomorphisms and Weyl rescalings that preserve $h_{ab} = \eta_{ab}$
- recall diffeom. in more detail:

$$\sigma^a \longrightarrow \sigma'^a = \sigma^a + \epsilon^a(\sigma)$$

$$h_{ab} \longrightarrow h'_{ab} = h_{cd} \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b}$$

$$\left(\frac{\partial \sigma'^a}{\partial \sigma^b} = \delta^a{}_b + \partial_b \epsilon^a \; ; \; \frac{\partial \sigma^a}{\partial \sigma'^b} = \delta^a{}_b - \partial_b \epsilon^a \right)$$

applying this specifically to η_{ab} gives

$$\eta_{ab} \longrightarrow \eta_{ab} - \eta_{ad} \partial_b \epsilon^d - \eta_{cb} \partial_a \epsilon^c \,, \quad \text{i.e.}$$

$$\delta h_{ab} = -(\partial_b \epsilon_a + \partial_a \epsilon_b)$$

- Weyl rescaling: $\eta_{ab} \longrightarrow (1+2\omega)\eta_{ab}$, i.e. $\delta h_{ab} = 2\omega\eta_{ab}$
- These two variations compensate if $\boxed{2\omega\eta_{ab} = \partial_a \epsilon_b + \partial_b \epsilon_a}$
- going to light-cone coordinates, this amounts to

1) $\partial^+ \epsilon^- + \partial^- \epsilon^+ = 2\omega\eta^{+-}$ ($\Rightarrow \omega$ fixed in terms of ϵ)

2) $\partial^- \epsilon^- = \partial^+ \epsilon^+ = 0$, i.e.

$$\partial_+ \epsilon^- = \partial_- \epsilon^+ = 0, \quad \text{i.e.} \quad \underline{\epsilon^- = \epsilon^-(\sigma^-)}, \ \underline{\epsilon^+ = \epsilon^+(\sigma^+)}$$

<u>Note:</u> This shows that the extra freedom corresponds to the reparameterizations of S^1 characterized by the Virasoro algebra.

- <u>Crucial new point:</u> instead of imposing $L_m = 0$ quantum-mechanically, let's fix this freedom classically (before quantization)
- finite version of our residual freedom:

$$\sigma^+ \longrightarrow \sigma'^+(\sigma^+) \ ; \ \sigma^- \longrightarrow \sigma'^-(\sigma^-) \ ;$$

in particular: $\tau' = \frac{1}{2}(\sigma'^+ + \sigma'^-)$ fulfills $\partial_+ \partial_- \tau' = 0$.

- Since the fields X^μ also fulfill this equation, we can choose, e.g., $\tau' \equiv (X^+(\tau,\sigma) - x^+)/p^+$

where $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$

$p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^{D-1})$ etc.

- Thus, in light cone gauge we have

$$\underline{X^+ = x^+ + p^+ \tau}, \quad \text{i.e.}$$

<u>no X^+-oscillators are excited.</u>

As a very useful implication, we have:

$$T_{ab} = 0 \Rightarrow (\dot{X} \pm X')^2 = 0 \Rightarrow$$

$$-2\underbrace{(\dot{X} \pm X')^+}_{\equiv p^+ \;!!} (\dot{X} \pm X')^- + (\dot{X} \pm X')^i (\dot{X} \pm X')^i = 0$$

i runs from 1 to $D-2$

(This is where the light cone gauge comes in.)

$$\Rightarrow (\dot{X} \pm X')^- = (\dot{X} \pm X')_\perp^2 / 2p^+ \; , \text{ i.e.}$$

not only X^+, but also X^- are in effect removed from the dynamics of the system.

- if we recall that $X^- = x^- + p^- \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^- e^{-in\tau} \cos n\sigma$ (where we have set $l = 1$),

then, at $\tau = 0$, the above expression for $(\dot{X} \pm X')^-$ implies

$$\boxed{\alpha_n^- = \frac{1}{p^+} \left(\frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \right)}$$

4.2 Light cone quantization

usual quantization procedure $\rightarrow$ canonical pairs of variables:

X^i, Π^i

x^-, p^+

(But not x^+, p^-, since p^- is just a function of the other variables.)

- The commutation relations read (directly in creator/annihilator form, where appropriate):

$$[\alpha_m^i, \alpha_n^j] = m\,\delta^{ij}\,\delta_{m+n}$$

$$[p^+, x^-] = i$$

- The quantum version of the definition of α_n^- reads

$$\alpha_n^- = \frac{1}{p^+}\Big(\frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\,\alpha_m^i: \; - a\,\delta_n\Big)$$

normal ordering — unknown constant (appears for the same reasons as earlier)

- Mass shell condition:

$$M^2 = 2p^+p^- - p_\perp^2 = 2(N-a) \quad \text{with} \quad N = \underbrace{\sum_{n=1}^{\infty} \alpha_{-n}^i\,\alpha_n^i}_{\text{now only transverse modes}}$$

(recall that $\ell = 1 \Leftrightarrow 2 = \frac{1}{\alpha'}$)

<u>important:</u> The mass shell condition is not a constraint on physical states but merely a consequence of the definition of p^-.

<u>Fock space</u>:

<u>level 0:</u> $|0,p\rangle$; $M^2 = -2a$

<u>level 1:</u> $\alpha_{-1}^i|0,p\rangle$; $M^2 = 2(1-a)$

$\to$ vector with transverse components only ;
inconsistent with Lorentz symmetry unless
$M^2 = 0$, i.e. $\underline{\underline{a = 1}}$.

<u>level 2</u>: states $\sim \alpha_{-1}^i \alpha_{-1}^j |0,p\rangle$ and $\sim \alpha_{-2}^i |0,p\rangle$

$M^2 = 2(2-a)$

...

We can also attempt a <u>direct calculation of a</u>:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}^i \alpha_n^i = \frac{1}{2}\sum_n :\alpha_{-n}^i \alpha_n^i: + \underbrace{\frac{D-2}{2}\sum_{n=1}^{\infty} n}_{\equiv -a}$$

• interesting observation (using "Zeta-fct.-regularization"):

$\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$ (well-defined for $s > 1$) ;

analytic continuation gives $\zeta(-1) = -1/12$

$\Rightarrow a = \frac{D-2}{24} \Rightarrow$ since we need $a = 1$, this implies

$\underline{\underline{D = 26}}$.

<u>Aside</u>: <u>Field-theoretic derivation of the normal ordering constant (providing a better justification of the above Zeta-fct. formula)</u>

Closed-string zero point energy : $-1 + (-1) = -2$
(from L_0 & $\tilde{L}_0$)

where the action was

$$S = -\frac{T}{2}\int d^2\sigma (\partial X)^2 \; ; \quad \text{and the period was } \pi .$$

- to match with standard field theory, let
  - metric: $(- + \cdots +) \to (+ - \cdots -)$
  - $\sigma^a \to \sigma^a/\sqrt{T}$, $X^\mu \to X^\mu/\sqrt{T}$

giving an action

$$S = \int d^2\sigma \, \frac{1}{2}(\partial X)^2 \text{ with period } 2\pi R \equiv \frac{\pi}{\sqrt{T}}, \text{ i.e.}$$

"compactified" on an S^1 with radius R.

$(R = 1/2\sqrt{T})$

- In these conventions, we should find a vacuum energy $-2\sqrt{T} = -1/R$.

(from rescaling of "time" by $\sqrt{T}$)

- The standard field theory formula for the vacuum energy (or the eff. potential) is (1 scalar d.o.f.)

$$V = \frac{1}{2}\sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \ln\left(k^2 + \left(\frac{n}{R}\right)^2\right) \quad *$$

(discrete "momenta" associated with the compact dimension along the S^1)

*

$$\boxed{Z \sim \ln \int D\varphi \, e^{-\frac{1}{2}\varphi D^{-1}\varphi}} \sim \ln \det D^{-1} \sim \mathrm{tr} \ln D^{-1}$$

With $R^{-1} \equiv M_c$ we have

$$V = \int_0^{M_c^2} dM^2 \frac{\partial}{\partial M^2} \left(\dots \right)$$ see above

$$= \int_0^{M_c} 2M dM \cdot \frac{1}{2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \sum_{-\infty}^{\infty} \frac{n^2}{k^2 + (nM)^2}$$

- we can replace $n^2/(\dots)$ with $n^2/(\dots) - 1/M^2$ (since $\int d^d k \, k^\alpha = 0$ for all α in dimensional regularization) and evaluate the sum:

$$V = \int_0^{M_c} M dM \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \cdot \frac{k^2}{M^4} \cdot \frac{\coth(\pi|k|/M)}{|k|/M}$$

- for the same reason as above, we can replace $\coth(\dots)$ with $\coth(\dots) - 1$.
- after this, the k-integration is finite (since $\coth(x) - 1 \to 0$ exponentially for $x \to \infty$); we can set $d = 2$ and rescale k to find

$$V = \int_0^{M_c} M dM \int_0^{\infty} \frac{d|k|}{2\pi} \cdot \frac{|k|}{M\pi} \left(\coth(|k|) - 1\right) = \int_0^{M_c} \frac{M dM}{2\pi^2 M} \cdot \pi^2 \underbrace{\zeta(-1)}_{= -1/12}$$

$$V = -\frac{M_c}{24} = -\frac{1}{24 R}$$

$\Rightarrow$ We need 24 physical X^μ's to get the desired $-1/R$;

$\Rightarrow \underline{\underline{D = 26}}$.

4.3 Light cone quantization & Lorentz symmetry

Noether-Theorem: Symmetry $\leftrightarrow$ conserved quantity

(e.g.: Translations $\leftrightarrow$ P^μ
Lorentz-symm. $\leftrightarrow$ $J^{\mu\nu}$)

quantum mechanically:

$\left.\begin{array}{l} P^\mu \\ J^{\mu\nu} \end{array}\right\}$ generate $\left\{\begin{array}{l} \text{translations} \\ \text{(Lorentz)-rotations} \end{array}\right\}$ via commutator

Specifically:

- Noether-Th. $\rightarrow$ $P^\mu = T\int_0^\pi d\sigma\, \dot{X}^\mu = p^\mu$ ($\sim$ our familiar α_0^μ)
  - a translation of Hilbert space vector is produced by $\exp[i\epsilon_\mu p^\mu] = \mathbb{1} + i\epsilon_\mu p^\mu + \cdots$
  - correspondingly, a translation on operators is generated by δ_ϵ "$\hat{=}$" $i\epsilon_\mu p^\mu$ via the commutator:

$$\delta_\epsilon x^\mu = [i\epsilon_\nu p^\nu, x^\mu] = i\epsilon_\nu(-i\eta^{\mu\nu}) = \epsilon^\mu$$

- Noether-Th. $\longrightarrow$ $J^{\mu\nu} = T\int_0^\pi d\sigma\left(X^\mu\dot{X}^\nu - X^\nu\dot{X}^\mu\right)$

$$= x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n}\left(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu\right)$$

$$\equiv l^{\mu\nu} + \underbrace{E^{\mu\nu}}_{\text{"inner" angular momentum of open string}}$$

as before, the infinit. Lorentz rotation is

$$\delta_\epsilon X^\mu = \left[\tfrac{i}{2}\epsilon_{\nu\rho} J^{\nu\rho}, X^\mu\right]$$

$$= \tfrac{i}{2}\epsilon_{\nu\rho}\left\{[X^\nu p^\rho, X^\mu] - [\mu\leftrightarrow\nu]\right\} = \tfrac{1}{2}\epsilon_{\nu\rho}\left(X^\nu\eta^{\rho\mu} - X^\mu\eta^{\rho\nu}\right)$$

$$= -\epsilon^\mu{}_\nu X^\nu \quad \text{(indeed, infinit. rotation by } \epsilon)$$

<u>In particular, for the "light-cone" strings:</u>

- our gauge choice has distinguished X^{D-1} relative to the X^i $(i=1\cdots D-2)$ by incorporating it in X^+ and distinguishing X^+.
- is Lorentz-symm. still intact after the non-covariant quantization that followed?
- for example, J^{i-} mixes X^+ & X^i (as explained above) so that a violation could show up in this generator.
- <u>recall:</u> $[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho}J^{\nu\sigma} - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \rho\leftrightarrow\sigma\end{pmatrix}$

$$\Rightarrow [J^{i-}, J^{j-}] = 0$$

- an explicit calculation, using the algebra of the α's, p's & x's, exposes an <u>anomaly</u>:

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right)$$

with $\Delta_m = m\,\dfrac{26-D}{12} + \dfrac{1}{m}\left(\dfrac{D-26}{12} + 2(1-a)\right)$ (→problems)

$\Rightarrow$ $D = 26$ & $a = 1$ required!

Comment: The fact that Lorentz-symm. is non-trivial after light-cone quantization is also apparent from the spectrum: Since only transverse oscillators are excited, all states come in $SO(D-2)$ representations. While this is natural for the massless states, it is highly non-trivial that for massive states these representations can be combined into the full $SO(D-1)$ representations required.

4.4 The no ghost theorem

(only brief overview; see GSW for more details)

[an alternative title of this section would be "The relation between LCQ and OCQ"]

- first, return to OCQ and consider operators of the type

$$V(k,\tau) \equiv \, :e^{ik^\mu X_\mu(0,\tau)}: \qquad \text{(all at}$$

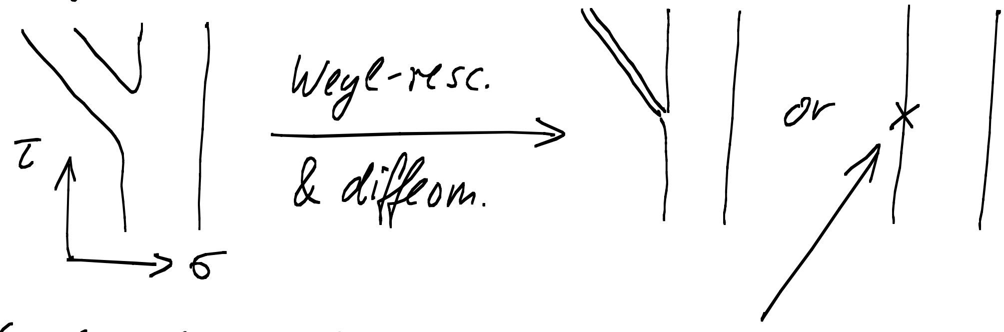
$$V_\zeta(k,\tau) \equiv \, :\zeta^\mu \dot{X}_\mu \, e^{ik^\mu X_\mu(0,\tau)}: \qquad \sigma = 0)$$

etc.

normal ordering

Aside: Although, at the moment, this is logically not necessary, it might be helpful to note these are Vertex operators

(i.e., they describe the emission or absorption of string states)



(a local operator at a certain τ and $\sigma = 0$ acts on a given state and produces a new state as it would arise after the emission/absorption of another state with momentum k)

<u>Note:</u>
- the exponent ensures that, if the old state had momentum p^μ, then the new state has momentum $p^\mu - k^\mu$.
- prefactor 1 $\longrightarrow$ tachyon
  - –"– $\quad \zeta \cdot X \longrightarrow$ photon with polarization ζ
  - $\vdots \qquad \vdots$ (other states)

- define $k^\mu_{(0)} = k^{(+,-,i)}_{(0)} = (0, -1, 0_\perp)$ to distinguish a particular (light-like) direction.
- focus on

$$V^i(k,\tau) \equiv \dot{X}^i(\tau)\, e^{ik\cdot X(\tau)}$$

with $k = n k_{(0)}$ and $i \in (1, \cdots, D-2)$ a transverse index and $n \in \mathbb{Z}$

(this corresponds to transverse photon emission/absorption, although we do not need this interpretation here.)

- we will <u>only</u> consider the action of the ops. on so-called "<u>allowed</u>" states, which have momenta

$$p^\mu = (1, p^-, 0_\perp) \quad (\text{i.e. } p\cdot k_{(0)} = 1 \ \& \ p_\perp = 0)$$

- observe: $\langle \text{allowed} | V^i(nk_{(0)}, \tau) | \text{allowed} \rangle$ is 2π-periodic in τ

$\Rightarrow$ a natural object to consider is its Fourier 0-mode:

$$\left\| \; A^i_n = \frac{1}{2\pi} \int_0^{2\pi} d\tau \, V^i(nk_{(0)}, \tau) \; \right\| \quad \text{DDF-ops.}$$

($\rightarrow$ Del Giudice, Di Vecchia, Fubini)

- Some important properties are:

  - $(A^i_n)^+ = A^i_{-n}$

  - $[N, A^i_n] = -n A^i_n \quad$ (i.e. $\hat{N} A^i_n |\psi\rangle = (N-n) A^i_n |\psi\rangle$ if $\hat{N}|\psi\rangle = N|\psi\rangle$)

  - $[L_m, A^i_n] = 0$

$\Rightarrow A^{i_1}_{-n_1} \cdots A^{i_m}_{-n_m} |0, p\rangle$ is physical and has $N = \sum_j n_j$

($|0,p\rangle$: allowed)

  - $[A^i_m, A^j_n] = m \delta_{ij} \delta_{m+n}$

- <u>Thus</u>: – The A_m^i algebra is identical to the α_m^i algebra of LCQ

(The states created by applying the A_m^i to the vaccum (= tachyon state) will be called <u>DDF states</u>)

(physical, positive metric subspace)

- to fully understand the OCQ phys. Fock space, we need to analyse the <u>orthogonal complement</u> to the DDF states (and see that there are <u>no neg. norm states</u> in it)

"no ghost theorem"

- <u>note</u>: it is sufficient to concentrate on the allowed states since all other states are related to them by Lorentz trfs.

- define $K_m \equiv k_{(0)} \cdot \alpha_m$ (K_m annihilates longit. "photons")

- some properties are:
  - $[K_m, K_n] = 0$
  - $[K_m, L_n] = m K_{m+n}$
  - if $|f\rangle$ is DDF state, then $K_m |f\rangle = 0$ for $m > 0$

(<u>this is easy to see</u>: A^i contains α^0- and α^{D-1}-creation ops. only in the combination $k_{(0)} \cdot \alpha_{-n}$, yet $[k_{(0)} \cdot \alpha_m, k_{(0)} \cdot \alpha_{-n}] \sim k_{(0)}^2 = 0$.)

- consider the states

$$|\{\lambda,\mu\},f\rangle = L_{-1}^{\lambda_1}\cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1}\cdots K_{-n}^{\mu_n}|f\rangle \qquad (*)$$

makes the state spurious — longit. components — DDF ("transverse state")

"Lemma": The states of $(*)$ with $|f\rangle$ running over a basis of DDF states and $\{\lambda,\mu\}$ over all "strings" of natural numbers are linearly indep.

idea of proof: for every $P = \sum r\cdot\lambda_r + \sum s\cdot\mu_s$ consider

$$\det \mathcal{M}^P_{\{\lambda,\mu\},\{\lambda',\mu'\}} = \det\left(\left(\langle f|\{\lambda,\mu\}\right)\left(\{\lambda',\mu'\}|f\rangle\right)\right)$$

and show that it is non-zero ...

"Lemma": The states of $(*)$ are as numerous (level by level) as the whole Fock space (proof by "counting"...)

$\Rightarrow$ They form a basis.

$\Rightarrow$ Any state can be decomposed in the basis of $(*)$ and written as $|\phi\rangle = |k\rangle + |s\rangle$

$|k\rangle$: $K_{-1}^{\mu_1}\cdots K_{-n}^{\mu_n}|f\rangle$

$|s\rangle$: $\underbrace{\text{the same, but with non-trivial string of } L\text{'s.}}_{\text{by definition spurious}}$

(since $\langle phys|L_{-n}\cdots|\cdots\rangle = 0$ for $n>0$)

"Lemma": $|\phi\rangle$ phys. $\Rightarrow$ $|k\rangle$ phys. & $|s\rangle$ phys.

(proof uses the fact that any spurious state can be written as $L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle$; the detailed calculation uses $[L_2, L_{-2}]$ (which involves D) and relies on $\underline{\underline{D=26}}$.)

- the above Lemma implies : $|\phi\rangle$ phys. $\Rightarrow$ $|s\rangle$ is null
- write $|k\rangle = |f\rangle + \underbrace{k_{-1}^{\mu_1} \cdots k_{-n}^{\mu_n}}_{\text{where not all } \mu\text{'s are zero}} |f'\rangle \equiv |f\rangle + |\tilde{k}\rangle$

"Lemma": $|k\rangle$ phys. $\Rightarrow$ $|\tilde{k}\rangle = 0$

$\Rightarrow$ finally: Any phys. state can be written as

$$|\phi\rangle = |f\rangle + |s\rangle \quad \text{(i.e. as "DDF" + "spurious")}$$

$\Rightarrow$ **no ghosts**, since $\langle s|f\rangle = \langle s|s\rangle = 0$.

Comment on identification of LCQ & OCQ Fock spaces:

LCQ		OCQ, DDF states
states with $p = (p^+, p^-, p_\perp)$	$\xleftrightarrow{\text{identified}}$	DDF states with this momentum p being allowed
$\downarrow$ all states with other p's by Lorentz-trf. (not obvious)		$\downarrow$ all other allowed p's by Lorentz-trf. (obvious!)
p'	$\xleftrightarrow{\text{identified}}$	p' allowed