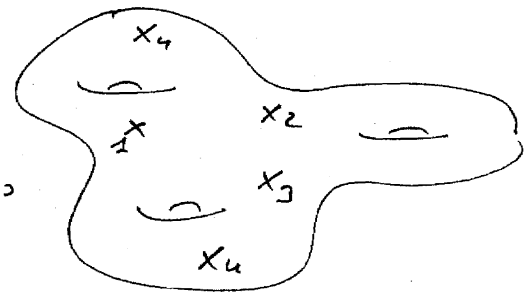
String Amplitudes

The interaction of n incoming/outgoing states can be written as

$$A = \sum_{\text{topologies}}$$



$$= \sum_{\text{topologies}} \int \frac{[dx][dg]}{V_{diff \times Weyl}} e^{-S_x} V_1 \cdots V_n$$

where the denominator $V_{d\times w}$ keeps into account the gauge invariance that must be modded out, and V_i is the vertex operator of the i-th in/out coming state.

We can avoid the denominator and properly quantize the theory by using the Faddeev Popov approach

Given a "preferred" metric $\hat{g}$ (flat metric) we know that there exists always a transformation ξ in Diff x Weyl such that, for each possible metric g, $g = \hat{g}^\xi = \xi(\hat{g})$ <u>locally</u>

This allows us to write

$$1 = \Delta_{FP}(g) \int d\xi \, \delta(g - \hat{g}^\xi)$$

and to replace it into the amplitude

$$A|_{\substack{\text{given} \\ \text{topology}}} = \int \frac{[dx\, dg\, d\xi]}{V_{d\times w}} \delta(g - \hat{g}^\xi) e^{-S_x} V_1 - V_n \Delta_{FP}(g)$$

We can then integrate in dg

$$A_{g.t.} = \int \frac{[dx\, d\xi]}{V_{diff} \times Weyl} e^{-S_x[\hat{g}^\xi]} V_1 - V_n \, \Delta_{FP}(\hat{g}^\xi)$$

moreover, since S_x is ξ-invariant, and so is Δ_{FP} (check)

$$A_{g.t.} = \int [dx]\, e^{-S_x[\hat{g}]} V_1 - V_n \, \Delta_{FP}(\hat{g})$$

$$\times \underbrace{\int \frac{d\xi}{V_{diff} \times Weyl}}_{=1}$$

<u>Subtleties</u> In order that the "change" of variable $dg = d\xi \left(\frac{\partial g}{\partial \xi}\right)$, of which $\frac{\partial g}{\partial \xi} \sim \Delta_{FP}$, be well defined we need a one-to-one map, i.e.

$$\forall g \; \exists! \, \xi \mid \hat{g}^\xi = g \quad \underline{\text{globally}}$$

1) <u>Local → Global</u>

$\Rightarrow \exists$ in general $g, g' \mid$

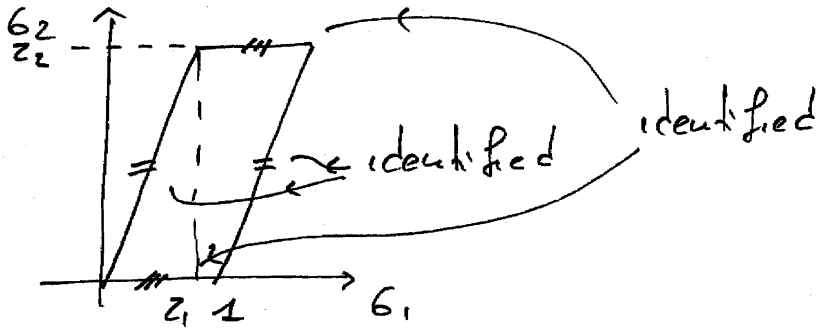
$\nexists \xi \mid g^\xi = g'$

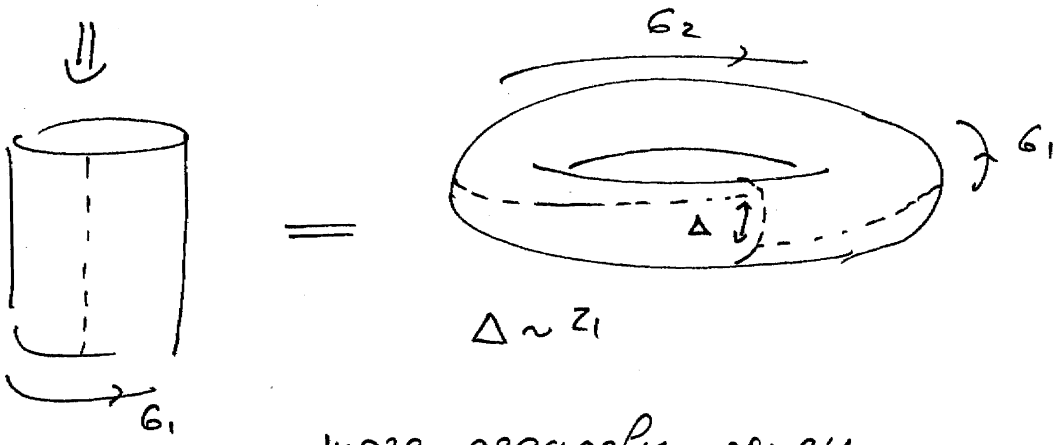
2) $\underline{\underline{\exists!}}$ $\Rightarrow \exists$ in general $\xi \mid g^\xi = g$

ξ is called conformal Killing vector and is part of the " " group CKG

Torus case

(1) It is always possible to bring g to $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ via Diff × Weyl but the boundary conditions are then non trivial





more precisely, given

$$z = \sigma_1 + i\sigma_2$$

$$z \sim z + (m + nz)$$

and the statement is that it is not possible to reabsorb z via Diff × Weyl without a modification

$$\eta_{ab} \to g_{ab}(z)$$

Choice Toller trivial b.c.

$\Rightarrow (\sigma_1, \sigma_2) \sim (\sigma_1 + m + z_1 n, \sigma_2 + z_2 n)$

let $\sigma_1' = \sigma_1 - \sigma_2 \frac{z_1}{z_2}$

$\sigma_2' = \frac{\sigma_2}{z_2}$

$(\sigma_1', \sigma_2') \sim (\sigma_1' + m + z_1 n - z_1 n, \sigma_2' + n)$

$\Rightarrow$ cvd

but $ds^2 = d\sigma_1^2 + d\sigma_2^2 = |d\sigma_1' + d\sigma_2' z|^2$

$= g_{ab} d\sigma'^a d\sigma'^b \qquad g = \begin{pmatrix} 1 & z_1 \\ z_1 & |z|^2 \end{pmatrix}$

Consequences

A single fiducial metric $\hat{g}$ is no more sufficient since $\mathcal{O}_{\hat{g}} = \{ \hat{g}^\xi, \xi \in \text{diff-weil} \}$

$\mathcal{O}_{\hat{g}} \neq$ set of all metric

Statement $\hat{g}(z)$ is indeed sufficient

$\int dz \; \mathcal{O}_{\hat{g}(z)} =$ set of all metrics

Then Replace $1 = \Delta_{FP}(g) \int d\xi \, \delta(g - \hat{g}^\xi)$

with

$1 = \Delta_{FP}(g) \int dz \, \delta(g - \hat{g}^{\xi(z)})$

Generalization Torus $\to$ a single complex Teichmüller Modulus

Other manifolds $\to$ more real moduli

$1 = \Delta_{FP}(g) \int \prod_{i=1}^{n_e} dt^i \, \delta(g - \hat{g}^{\xi(t_i)})$

(2) $\xi: \sigma^a \longmapsto \sigma^a + \alpha^a$

is a perfectly well-defined Diff transformation

but $\xi: \eta_{\mu\nu} \longmapsto \eta_{\mu\nu}$

Statement: on the torus only these two transformations, parametrized by α^1 and α^2, defines the continuous part of the CKG

Solution: insert a σ-dependence in Δ_{FP}

$$1 = \Delta_{FP}[\sigma, g] \int \prod_i dt^i \int d\xi\, \delta(g - \hat{g}^\xi(t^i))\, \delta(\sigma_1 - \hat{\sigma}_1^\xi) \times \delta(\sigma_2 - \hat{\sigma}_2^\xi)$$

the problem is solved since, given

$\hat{\xi} \mid g = \hat{g}\hat{\xi}$, $\exists!\ \xi_{CKG} \mid \sigma = \hat{\sigma}^{\hat{\xi}\cdot\xi_{CKG}}$

Generalization

It is possible to reabsorb the ambiguity of a CKG of real dimension n_K by introducing n_K Dirac deltas

$\delta(\sigma_i - \hat{\sigma}_i^\xi)$

in the definition of Δ_{FP}

$$1 = \Delta_{FP}[\sigma_i, g] \int_F \prod_{i=1}^{n_K} dt^i \int d\xi\, \delta(g - \hat{g}^\xi) \prod_{i=1}^{n_K} \delta(\sigma_i - \hat{\sigma}_i^\xi)$$

Integration region for the moduli

$\Rightarrow$ Torus case

In principle $F \subset \mathbb{R}^2$, let's see the details

take the parametrization

$$(\sigma_1, \sigma_2) \sim (\sigma_1 + m + \tau_1 n, \sigma_2 + \tau_2 n)$$

$$ds^2 = d\sigma_1^2 + d\sigma_2^2$$

def $\quad \omega = \sigma_1 + i\sigma_2$

$\Rightarrow \quad ds^2 = d\omega\, d\bar{\omega}$

$$\omega \sim \omega + m + \tau n$$

Now (1) replace τ with $\tau + 1$ $\quad (T)$

$\Rightarrow$ same b.c. in $m' = m + n$, $n' = n$

(2) replace τ with $-1/\tau$ $\quad (S)$

$$\omega \sim \omega + m - \frac{n}{\tau}$$

redefine $\omega' = \tau\omega$

$\rightarrow \quad \omega' \sim \omega' + m\tau - n \Rightarrow$ same as before

$(ds^2 = d\omega' d\bar{\omega}'$ via Weyl rsc.$)$

$\Rightarrow \quad F = \mathbb{R}^2 / \text{Group Generated by } T \text{ and } S$

Statement ① T, S generate (span)

$Sl(2, \mathbb{Z})$ of transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z}$$

$$ad - bc = 1$$

② $F \Rightarrow \quad \mathrm{Re}\,\tau \in [-\tfrac{1}{2}, \tfrac{1}{2}]$

$|\tau| \geq 1 \quad (\tau_1 > 0)$

$$1 = \Delta_{FP}(g,G)\int\prod_{i=1}^{ne} dt^i \int d\xi\ \delta(g-\hat{g}(t)^\xi)\prod_{a=1}^{nu}\delta(G^a-\hat{G}^{a\xi})$$

$$\Delta_{FP}^{-1}(g,G) = \int\prod_{i=1}^{ne} dt \int d\xi\ \delta(g-\hat{g}(t)^\xi)\prod_{a=1}^{nu}\delta(G^a-\hat{G}^{a\xi})$$

now $\exists!$ zero but (last subtlety)

notice that we fixed all continuous symmetries; $\exists$ a last discrete ch. symm.

TORUS $\quad G^a \to -G^a$

To fix it as it is enough to remember that there are $n_R = 2$ zeros, and to expand around just one of them

$$\Delta_{FP}^{-1}(g,G) = n_R\int\prod_{i=1}^{ne} dt^i d\xi\ \delta(g-\hat{g}(t)^\xi)\prod_{a=1}^{nu}\delta(G^a-\hat{G}^{a\xi})$$

$$d\xi = d\omega\, d\epsilon$$

$$g-\hat{g}(t)^\xi = \delta g\Big|_{\sim \text{zero}} = (2\omega - D_c\epsilon^c)\hat{g}_{ab} - (D_a\epsilon_b + D_b\epsilon_a - 2\epsilon^c)\hat{g}_{ab} + t^i\partial_{t^i}\hat{g}_{ab}$$

$$\Delta_{FP}^{-1}(g,G) = n_R\int\prod_{i=1}^{ne} dt\, d\omega\, d\epsilon\ \delta\left((2\omega - D_c\epsilon^c)\hat{g}_{ab} - 2P\epsilon_{ab} + t^i\partial_{t^i}\hat{g}_{ab}\right)\prod_{a=1}^{2}\delta(\epsilon^a)$$

$\Leftarrow$ Specific for the Torus

exponentiate the δ's

$$\Delta_{FP}^{-1}(\hat{g},\hat{G}) = n_R\int\prod_{i=1}^{ne} dt^i d\omega\, d\epsilon^a\, d\beta^{ab}\, dx_a$$

$$\exp\left\{2\pi i\int d^2\sigma\sqrt{g}\,\beta^{ab}\left[(2\omega - D_c\epsilon^c)\mathbb{1}_{ab} + 2P\epsilon_{ab} + t^i\partial_{t^i}\hat{g}_{ab}\right]\right\}$$

$$\exp\left\{2\pi i\,\epsilon^a x_a\right\}$$

Integrate $d\omega$ $\to$ β^{ab} traceless $(\beta^a{}_a = 0)$

$$\Delta_{FP}^{-1}(\hat{g}, \hat{\sigma}) = n_R \prod_{i=1}^{n_\ell} \int dt^i \, d\epsilon^a \, d\beta^{ab} \, dx_a$$

$$\exp\left\{ 2\pi i \int d^2\sigma \sqrt{\hat{g}} \; \beta^{ab} \left[(2P\epsilon)_{ab} - t^i \partial_{t^i} g \right] \right\}$$

$$\exp\left\{ 2\pi i \epsilon \cdot x \right)$$

$\to$ It is just a determinant of a new modified operator

$\Rightarrow$ Pass to Grassmann variables to "invert" it

$$\epsilon^a \to c^a$$
$$\beta_{ab} \to b_{ab}$$
$$x \to \eta$$
$$t \to \xi$$

$$\Delta_{FP}(\hat{g}, \hat{\sigma}) = \frac{1}{n_R} \int [db\,dc] \, d\xi^i \, d\eta_a$$
$$e^{-\int d^2\sigma \sqrt{g} \left[b^{ab} (Pc)_{ab} + \xi^i \partial_{t^i} g) \right]}$$
$$e^{\eta_a c^a(\hat{\sigma})}$$

integrate in ξ and η

$$\Delta_{FP}(\hat{g}, \hat{\sigma}) = \frac{1}{n_R} \int [db\,dc] \, e^{-S^{ghost}} \times \prod_{i=1}^{n_\ell} \left[\int d^2\sigma \sqrt{g} \; \underline{b^{ab} \partial_{t^i} g_{ab}} \right]$$
$$\times \prod_{a=1}^{2} \underline{c^a(\hat{\sigma})}$$

Torus amplitude

$$A_{torus} = \frac{\int [dx][dg]}{V_{diff \times Weyl}} \int_F d\tau_1 d\tau_2 \, \Delta_{FP}(\sigma, g)$$

$$\times \int d\xi \, \delta(g - \hat{g}^{\xi}(\tau)) \, \delta(\sigma_1 - \hat{\sigma}_1^{\xi}) \, \delta(\sigma_2 - \hat{\sigma}_2^{\xi}) \, e^{-S_x[g]} \, V_1 - V_n$$

This amplitude is meaningless unless an explicit σ_1, σ_2 integration is explicitated.

In the torus case, since the volume of the CKG is finite and equal to the volume of the torus itself, it is possible to introduce an integration of the form

$$1 = \frac{\int d^2\sigma}{V_{TORUS}}$$

In other words, one fixes the CKG problem by just dividing away $V_{CKG} = V_{TORUS}$?

No $\rightarrow$ B-insertions in Δ_{FP}!

In the general case, knowing $V_i = \int d^2\sigma \sqrt{g} \, V_i(\sigma)$, we can identify the σ-dependence in Δ_{FP} with the σ-dep. in $n_k/2$ vertex operators.

The CKG symmetry is fixed by fixing points on the manifold + including insertions in the path integral.

In the torus case, replacing the Δ_{FP}

$$A_{torus} = \frac{1}{n_R} \int \frac{dz_1 dz_2}{V_{TORUS}} \int [dx]\,[db\,dc] \overbrace{\int d^2\sigma \sqrt{\hat{g}}\,(b^{ab})_{z_1} g_{ab}}^{B_1} \times$$

$$\times \overbrace{\int d^2\sigma \sqrt{\hat{g}}\,(b^{ab})_{z_2} g_{ab}}^{B_2}\; c^1(\hat{\sigma})\, c^2(\hat{\sigma})\; e^{-[S_X + S_{ghosts}]}$$

$n_R = 2$

$= z_2$

$V_1 - V_M$

Partition function

$$Z_{torus} = \frac{1}{2} \int_F \frac{dz_1 dz_2}{z_2} \langle B_1 B_2 c^1 c^2 \rangle_{ghosts} \times \langle 1 \rangle_{x\text{-fields}}$$

$\Rightarrow$ Relation with Coleman-Weinberg formula

$\Rightarrow$ Rôle of the ghost partition function

$\Rightarrow$ Self-regulation of UV divergences