8 CFT - I

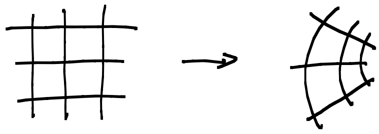
(For a "proper" introduction see Blumenhagen/Plauschinn or the "bible": Di Francesco/Mathieu/Senechal)

8.1 Conformal trfs. in general

• **Conf. trf.** $\equiv$ diffeomorphism, under which the metric changes only by an overall factor, i.e.:

$$x \to x' = x'(x) \implies g_{\mu\nu} \to g'_{\mu\nu}\,, \text{ with } g'_{\mu\nu}(x') = \underset{\substack{\uparrow \\ \text{generic diff.}}}{g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}} = \underset{\substack{\uparrow \\ \text{non-trivial restriction} \\ (\text{for } d \geq 2)}}{g_{\mu\nu}(x)\, e^{2\omega(x)}}$$

• Intuitively: angle-preserving transformations, i.e.



etc.

• Infinitesimally: $\delta x^\mu = \epsilon^\mu$; let $g_{\mu\nu} = \eta_{\mu\nu}$

$\implies$ the "non-trivial restriction" above reads $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega \eta_{\mu\nu}$ $\implies$ $(2-d)\partial_\mu \partial_\nu \omega = 0$

& $\partial^2 \omega = 0$

(cf. problems)

$\implies$ for $d > 2$, ω is a quadratic fct.; group of conf. trfs. has finitely many parameters.

• In detail, conf. trfs. in $\mathbb{R}^d$ are:

(1) Poincaré trfs. $\to \frac{d(d+1)}{2}$ param.s

(2) dilation: $x'^\mu = \alpha x^\mu \to 1$ parameter

(3) special conf. trfs: $\quad x'^\mu = \dfrac{x^\mu - x^2 b^\mu}{1 - 2b\cdot x + b^2 x^2} \to d$ params.

Can be realized as: "inversion $\circ$ translation $\circ$ inversion" (where inversion $\equiv \{x^\mu \to \frac{x^\mu}{x^2}\}$)

• It can be shown:

Conf. trfs. in $\mathbb{R}^d \equiv SO(1, d+1)$; Conf. trfs. in $\mathbb{R}^{1,d-1} \equiv SO(2, d)$

8.2 Conf. trfs. in $d=2$

- For $d-2=0$, the constraint $\partial_a \partial_b \omega = 0$ does not follow.
- We need to start again from the basic constraint

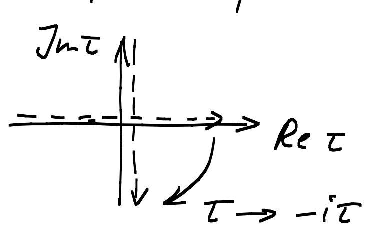
$$\partial_a \epsilon_b + \partial_b \epsilon_a = 2\omega \eta_{ab} .$$

- We already know that this is precisely the residual gauge freedom of the WS in flat gauge. (This is why we care so much about CFTs!)

- It is solved by: $\epsilon^+ = \epsilon^+(\sigma^+)$; $\epsilon^- = \epsilon^-(\sigma^-)$

  [2 arbitrary fcts. ; hence infinitely many parameters ; ω defined by $\omega = \partial_+ \epsilon_- + \partial_- \epsilon_+$]

- As is well-known from QFT, it is often convenient to rotate the integration contour



, to get a euclidean theory. ("Wick rotation")

τ only appears as an integration variable in $\int DX\,Db\,Dc\, e^{i\int d\tau d\sigma \mathcal{L}}$.

Hence the contour deformation is allowed if the integral falls off sufficiently quickly at large $|\tau|$. This is hard to justify rigorously in the present context. We simply assume that the relevant fcts X, b, c are such that Wick rotation is allowed.

- The effect is $\dot{X}^2 \to -\dot{X}^2$; $\int_{-\infty}^{\infty} d\tau \to -i \int_{-\infty}^{\infty} d\tau$; $e^{iS} \to e^{-S_E}$

  pos. definite action with metric δ_{ab} (instead of η_{ab}).

- Our 2d FT is now defined on the euclidean plane $\mathbb{R}^2$, which we parametrize by $z, \bar{z}$.

- $ds^2 = \frac{1}{2} dz\, d\bar{z}$ (with $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$ formally independent) is the euclidean metric.

$$\left(h_{z\bar z} = \tfrac{1}{2} \ , \ h_{zz} = h_{\bar z \bar z} = 0 \right)$$

- $\partial_a \epsilon^b + \partial_b \epsilon^a = 2\omega \delta_a{}^b \quad \Rightarrow \quad \partial_{\bar z} \epsilon^z = 0 \ \Rightarrow \ \epsilon^z = \epsilon^z(z)$ ← holomorphic!

  with $a, b = z, \bar z$

- Since $\left\{ \begin{array}{l} \epsilon^z = \epsilon^1 + i\epsilon^2 \\ \epsilon^{\bar z} = \epsilon^1 - i \epsilon^2 \end{array} \right\}$, we know that $\epsilon^{\bar z} = \overline{\epsilon^z(z)}$.

- We define $\epsilon(z) \equiv \epsilon^z(\bar z)$.

<u>Summary:</u> In (euclidean) $\mathbb{R}^2$ and using $\mathbb{R}^2 \equiv \mathbb{C}$, conformal trfs. are <u>holomorphic trfs.</u>, i.e.

$$z \to z + \epsilon(z) \ ; \ \bar z \to \bar z + \bar\epsilon(\bar z)$$

(or, for finite trfs., $z \to z' = w(z)$; $\bar z \to \bar z' = \bar w(\bar z)$)

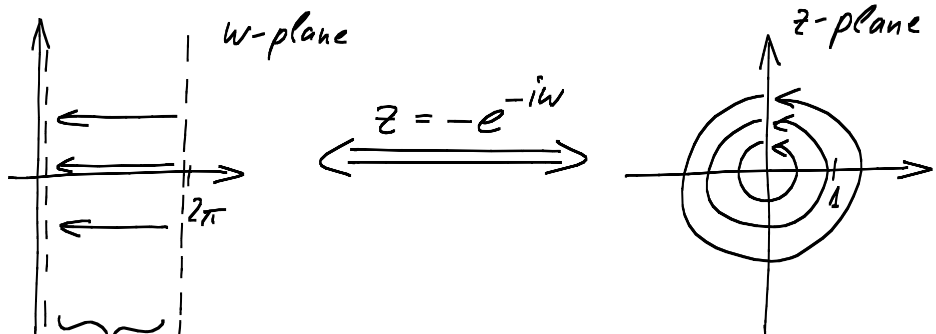
- A generic fct. (scalar field) $\phi = \phi(z, \bar z)$ transforms as

$$\delta\phi = -\epsilon^a \partial_a \phi = \left(-\epsilon^z \partial_z + \epsilon^{\bar z} \partial_{\bar z}\right)\phi \equiv \left(-\epsilon\partial + \bar\epsilon \bar\partial\right)\phi$$

- With the Laurent expansion $\epsilon = \sum_{n=-\infty}^{\infty} c_n z^n$ we get

$$\delta\phi = \sum_{-\infty}^{\infty} \left(c_{n+1} \ell_n + \bar c_{n+1} \bar\ell_n \right)\phi \ ; \ \underbrace{\begin{array}{l} \ell_n \equiv -z^{n+1}\partial \\ \bar\ell_n \equiv -\bar z^{n+1} \bar\partial \end{array}}_{\text{two independent sets of Virasoro generators (check!)}}$$

- <u>A particularly important trf.</u>



"string width 2π" (now more convenient)

In the second parametrization, time goes in the radial direction while "σ" corresponds to the phase.

8.3 String world-sheet CFT

- A CFT is a FT invariant under conf. trfs.

  Comments:
  - Here we view our FT as a FT with fixed metric (the metric does not transform). The trf. $x \to x'$ is viewed not as a reparametrization, but as an "active" symm. trf., i.e.

    $$[\phi : x \mapsto \phi(x)] \longrightarrow [\phi' : x' \mapsto \phi'(x') \equiv \phi(x)]$$

  - Due to the large conf. group in $d=2$, a lot can be said without an action. Especially in this case "invariance" means "invarians of the correlation fcts" $\langle \phi(x_1)\phi(x_2)\cdots\rangle$.
  - Intuitively speaking, CFTs are "scale invariant" (i.e. the physics looks the same at any length scale)

- The WS-FT after gauge fixing is (at least classically) a CFT since
  - If we apply a conf. trf. and change the metric appropriately S is invariant (diff.)
  - The "appropriate" metric change is a rescaling and can hence be undone by a Weyl-trf. (under which S is again invariant.)
  - Hence, in total, S is inv. under a conf. trf. with fixed metric.

    (Check this explicitly using $S \sim \int d^2z\, (\partial_z X)(\partial_{\bar z} X)$ and $z \to z'(z)$. Note: $\int d^2z \equiv \int dz\, d\bar z \equiv \int d\sigma^1 d\sigma^2$)

- By calculations completely analogous to those in σ^+, σ^- in Mink. space we have in $z, \bar z$ in eucl. space:

$$T_{ab} \text{ traceless} \quad \longrightarrow \quad T_{z\bar z} = 0$$

$$\text{conserved} \quad \longrightarrow \quad \partial_{\bar z} T_{zz} = \partial_z T_{\bar z \bar z} = 0$$

- Define $T \equiv T_{zz}$ and $\bar{T} \equiv T_{\bar{z}\bar{z}}$ (but these are independent!)
- The most important examples for now are

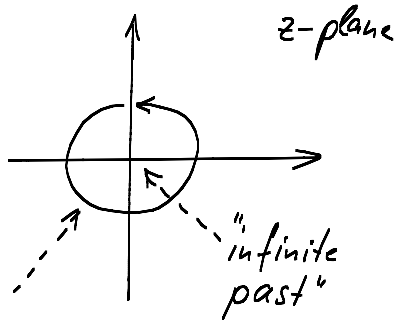
$$S = \frac{1}{2\pi} \int d^2z \, \partial X \bar{\partial} X \qquad \text{and} \qquad S = \frac{1}{2\pi} \int d^2 z \, b \bar{\partial} c$$

$$\downarrow$$

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}) \quad ; \quad T = -2 \partial X \partial X$$

$$\uparrow \text{independent!} \qquad \bar{T} = -2 \bar{\partial}\bar{X}\bar{\partial}\bar{X}.$$

- As before $L_m = \dfrac{1}{2\pi i} \oint dz \, z^{m+1} \, T(z)$.



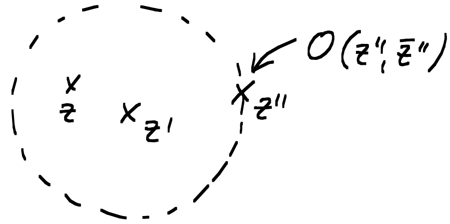
Here we work in the "z-frame" and this contour corresponds to integrating "once across the cylinder" of the string.

8.4 Operator product expansion

- Let $\hat{O}_i(z,\bar{z})$ be the set of all local operators (X, ∂X, $X \cdot X$ etc.)
- The "OPE" states that

$$\hat{O}_i(z,\bar{z}) \, \hat{O}_j(z',\bar{z}') = \sum_k C_{ij}^{\;\;k}(z-z', \bar{z}-\bar{z}') \, \hat{O}_k(z',\bar{z}') \,,$$

crucially, in 2d CFT's, with finite radius of convergence (equal to distance to next operator).



- Here and in what follows, such expressions will always mean "inserted in $\langle \cdots \quad \rangle$"

  (with any other operators here you want)

- Technically, this usually means "under the path integral" or, in operator language, time-ordered (w-frame) / radial-ordered (z-frame) and inside $\langle \cdots \rangle$.

- A proof can be given (for X-CFT or even in general, → Polch.), but view the above as a sufficiently "natural" generalization of the Laurant-expansion to "matrices" and won't bother.

8.5 Ward identities

- Consider a global symm. trf. $\phi \to \phi' = \phi + \epsilon\delta\phi$ (i.e. $S[\phi'] = S[\phi]$ & $\mathcal{D}\phi' = \mathcal{D}\phi$).

- Now let $\epsilon = \epsilon(x)$, which in general implies $\delta S \neq 0$. However, $\delta S = 0$ if $\epsilon = \text{const.}$, which implies $\delta S \sim \partial_a \epsilon$. This logic is conventionally used to define a conserved current:

$$S[\phi'] - S[\phi] = \delta S[\phi] \equiv \underset{\substack{\uparrow \\ \text{"stringy" convention}}}{\frac{1}{2\pi}} \int J^a \partial_a \epsilon$$

"stringy" convention (here & below we follow the notation of D. Tong)

- Now we find:

$$\int \mathcal{D}\phi\, e^{-S[\phi]} \underset{\uparrow}{=} \int \mathcal{D}\phi'\, e^{-S[\phi']} = \int \mathcal{D}\phi\, e^{-S[\phi] - \delta S[\phi]} = \cdots$$

follows by viewing $\phi \to \phi'$ as a renaming of an integration variable

fct. ϵ is arbitrary

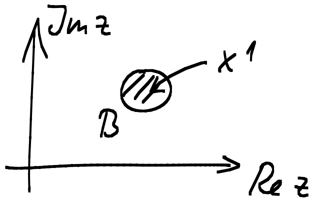
$$\cdots = \int \mathcal{D}\phi \left(1 - \frac{1}{2\pi}\int d^dx\, J^a(x)\,\partial_a \epsilon(x)\right) e^{-S[\phi]} \Rightarrow \underbrace{\langle \partial_a J^a \rangle = 0}_{\text{current conservation in the quantum theory}}$$

- Now let us redo the analysis, but with a specific region ("ball") B where $\epsilon \neq 0$ and an operator insertion at x_1:

$$\Rightarrow \int \mathcal{D}\phi \left(1 - \frac{1}{2\pi}\int_B d^dx\, J^a \partial_a \epsilon(x)\right)\left(\mathcal{O}(x_1) + \epsilon(x_1)\delta\mathcal{O}(x_1)\right) e^{-S} = \int \mathcal{D}\phi\, \mathcal{O}\, e^{-S}$$

$$\underset{\uparrow}{\Longrightarrow} \quad \boxed{\langle \delta\mathcal{O}(x_1)\cdots\rangle = -\frac{1}{2\pi}\int_B d^dx\, \partial_a \langle J^a(x)\mathcal{O}(x_1)\cdots\rangle}$$

at leading order in ϵ, assuming $\epsilon = \text{const.}$ inside B



- Clearly, the above is non-trivial only if $x^1 \in \mathcal{B}$ ("..." denotes other operators outside $\mathcal{B}$).

- Specifically for $d=2$ (see figure) we have:

$$\int_{\mathcal{B}} d^2z\, \partial_a J^a \cdots = \int_{\partial\mathcal{B}} \hat{n}^a J_a \cdots = \oint (\vec{J} \times d\vec{z})_3 = \oint (J_1 dx^2 - J_2 dx^1) = \cdots$$

$$dz = dx^1 + i dx^2 \quad ; \quad J_z = \tfrac{1}{2}(J_1 - iJ_2)$$

$$\cdots = -i \oint \left(dz\, J_z \cdots - d\bar{z}\, J_{\bar{z}} \cdots \right)$$

$$\Rightarrow \boxed{\delta O(z_1) = \frac{i}{2\pi} \oint \left(dz\, J_z(z,\bar{z}) - d\bar{z}\, J_{\bar{z}}(z,\bar{z}) \right) O(z_1)}$$

(close enough to z_1 so that no other insertions are inside contour)

(always under path int. with further insertions)

8.6 Conformal Ward identity

- Now we apply this to conf. trfs., $\delta z = \epsilon(z)$, treated as global symm. (i.e. one symm. for each indep. holom. fct. $\epsilon(z)$).

$$\delta S = \int d^2\sigma \frac{\delta S}{\delta h_{ab}} \delta h_{ab} = \int d^2\sigma \left(-\frac{\sqrt{h}}{4\pi} T^{ab} \right) \delta h_{ab}$$

Since δS arises only because we don't change the metric in a conf. trf.

$$= -\frac{1}{4\pi} \int d^2\sigma\, T^{ab} (-\partial_a \epsilon_b - \partial_b \epsilon_a) = \frac{1}{2\pi} \int d^2\sigma\, T_{ab} \partial^a \epsilon^b$$

$$= \frac{1}{2\pi} \int d^2z \left(T_{zz}\, \partial^z \delta z + T_{\bar{z}\bar{z}}\, \partial^{\bar{z}} \delta\bar{z} \right) = \frac{1}{\pi} \int d^2z \left(T_{zz}\, \partial_{\bar{z}} \epsilon + T_{\bar{z}\bar{z}}\, \partial_z \bar{\epsilon} \right).$$

$T_{z\bar{z}} = T_{\bar{z}z} = 0$ (T traceless)

- This is automatically zero if ϵ is holomorphic. Making our trf.

local instead of global ($\epsilon \to \epsilon(x)$) corresponds to
$\epsilon(z) \to \epsilon(z) f(\bar{z})$, where $f(\bar{z})$ plays the role of $\epsilon(x)$ of "8.5".

$$\Rightarrow \delta S = \frac{1}{2\pi}\int d^2z \Big[\underbrace{2T_{zz}(z)\epsilon(z)\,\partial_{\bar{z}} f(\bar{z})}_{\hat{=}\; J^a \partial_a \epsilon \text{ of "8.6"}} + \dots\Big]$$

$$\Rightarrow J^{\bar{z}} = 2T_{zz}(z)\epsilon(z) \equiv 2T(z)\epsilon(z) \quad ; \quad \boxed{J_z = T(z)\epsilon(z)}$$

(and analogously $J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$)

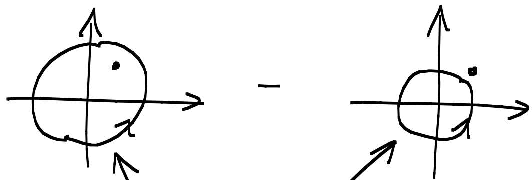
- Plugging this into our formula "$\delta O = \oint \cdots$", we see that the integration just picks up the residues:

$$\boxed{\delta O(z_1) = -\operatorname{Res}_{z\to z_1}\left[\epsilon(z)T(z)O(z_1)\right] - \operatorname{Res}_{\bar{z}\to\bar{z}_1}\left[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})O(z_1)\right]}$$

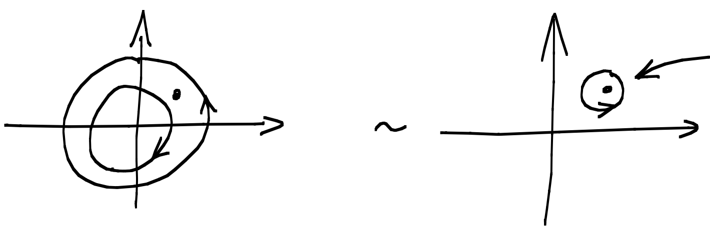
This crucial result tells us how to obtain the (full, quantum!) trf. properties of any O from its OPE with T_{ab}.

- Comment: A slightly different set of arguments leading to the same conclusion goes roughly as follows (→ Polch.):

$$\delta O \sim [\underset{\substack{\uparrow \\ \text{symm. generator}}}{Q}, O] \sim QO - OQ \sim$$



"$\oint$" of the current, producing Q after/befor O in radial quantization

$\sim$ 

integral of current J on contour around O, just as was found before!