9 CFT - II

9.1 Primary operators

- Def: An operator $O(z,\bar z)$ is called primary if it transforms according to

$$O(z,\bar z) \longrightarrow O'(z',\bar z') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar z'}{\partial \bar z}\right)^{-\bar h} O(z,\bar z).$$

$(h,\bar h)$ are called weights.

- Quite generally, the whole set of operators (and the whole Hilbert space) can be constructed from the primaries (hence the name). In statistical applications, the weights correspond to critical exponents. For us, the primaries will be important because the vertex operators will be a special subset of primaries.

- Intuitively, one can understand the primaries as follows:
  - any point (and hence any local operator) can be mapped to $z=0$.
  - focus on $z \to e^{\omega + i\varphi} z$
  - it is now obvious that $s = h - \bar h$ measures the spin, while $\Delta = h + \bar h$ measures the scaling dimension of O.
  - thus, one effectively considers the operators which are "eigenvectors" of rotation & dilatation. One could also say that this is a "tensor basis" of operators.

- It is easy to check that, infinitesimally,

$$\delta O(z,\bar z) = -\left(h\partial_z \epsilon + \bar h \partial_{\bar z}\bar\epsilon + \epsilon\partial_z + \bar\epsilon\partial_{\bar z}\right) O(z,\bar z)$$

- Comparing this with the TO & $\bar T O$ - OPE, one easily derives:

$$T(z)\,O(w,\bar w) = h\,\frac{O(w,\bar w)}{(z-w)^2} + \frac{\partial O(w,\bar w)}{z-w} + \text{non-singular}$$

$$\bar T(\bar z)\,O(w,\bar w) = \bar h\,\frac{O(w,\bar w)}{(\bar z-\bar w)^2} + \frac{\bar\partial O(w,\bar w)}{\bar z-\bar w} + \text{non-singular}$$

$\longrightarrow$ problems.

9.2 Our most important example: The free boson

- $S = \frac{1}{4\pi\alpha'} \int d^2z \, \partial_a X \partial^a X$ (at the moment, one X is enough)

- propagator: $\langle X(z,\bar z) X(z',\bar z')\rangle = \int DX \, e^{-S} X(z,\bar z) X(z',\bar z')$

$$= -\frac{\alpha'}{2} \ln |z-z'|^2$$

This can be checked in many ways, e.g. by remembering that the propagator is the inverse of the differential operator appearing in the free action:

$$-\frac{1}{2\pi\alpha'} \partial^2 \langle X(z,\bar z) X(z',\bar z')\rangle = \delta^2(z-z', \bar z - \bar z'),$$

Klein-Gordon, not $(\partial_z)^2$!

and solving this diff.eq..

One can also immediately see the log-behaviour typical of $d=2$ by observing:

$$G(x,x') \sim \int \frac{d^dk}{k^2} e^{ik(x-x')} \sim \frac{1}{d-2}\left[\frac{1}{(x-x')^2}\right]^{\frac{d-2}{2}}$$

(by dim. analysis or by rescaling k)

$$\Big\downarrow \; d\to 2$$

$$G(x,x') \sim \ln (x-x')^2$$

(by noting that $\frac{1}{\epsilon} x^\epsilon = \frac{1}{\epsilon} e^{\epsilon \ln x} \sim \ln x$ at small ϵ)

Detailed calculation: $\to$ problems

- We already know T_{ab} and it is easy to recall how we defined $T(z)$ and $\bar T(\bar z)$ and find:

$$T(z) = -\frac{1}{\alpha'} \partial X \partial X \; ; \; \bar T(\bar z) = -\frac{1}{\alpha'} \bar\partial X \bar\partial X$$

this is now ∂_z!

- Quantum mechanically, one of course has to normal order $\equiv$

$\equiv$ subtract the vacuum expectation value, i.e.

$$T = -\frac{1}{\alpha'} :\partial X \partial X: \equiv -\frac{1}{\alpha'} \lim_{z \to z'} \left(\partial X(z) \partial X(z') - \langle \partial X(z) \partial X(z') \rangle \right)$$

This def. is more natural for the path-integral approach. (It is also obvious that it "does the right thing" since $\langle T \rangle = 0$ is explicitly realized.)

- To determine the trf. properties of an operator, we need the OPE with T. Let's start with the operator X:

"contraction"

$$T(z) X(w,\bar{w}) = -\frac{1}{\alpha'} :\partial X(z) \partial X(z): X(w,\bar{w}) = -\frac{2}{\alpha'} \partial X(z) \partial X(z) X(w,\bar{w}) + \text{non.-sing.}$$

$$= -\frac{2}{\alpha'} \partial X(z) \partial_z \left(-\frac{\alpha'}{2} \ln (z-w)(\bar{z}-\bar{w}) \right) + \cdots$$

$$= \frac{\partial X(z)}{z-w} + \cdots = \frac{\partial X(w)}{z-w} + \cdots$$

- The "contraction" (i.e. the replacement of a pair $X(z_1)X(z_2)$ by the propagator) arises in standard QFT:

time ordering of $(\cdots) = :(\cdots): + :(\cdots \text{all contractions} \cdots):$

$\boxed{\text{Wick-theorem}}$

[in our case, for obvious reasons, <u>without</u> contractions inside $T(z)$, since $T(z)$ is already normal-ordered]

- The singular part is always associated with the contractions between fields in $T(z)$ and in $O(w)$.

- Another way to argue for the appearance of contractions is directly through the path integral

$$\int D\phi \cdots \underbrace{\phi_1 \cdots \phi_2}_{\text{in } T(z)} \underbrace{\phi_3 \cdots \phi_4}_{\text{in } O(w)} \underbrace{\phi_5 \cdots \phi_6}_{\text{other}} e^{-S}$$

$\leftarrow$ evaluate as usual (with quadratic S) in terms of propagators

- Returning to our example, we see that $T(z)X(w,\bar{w})$ (and analogously $\bar{T}(\bar{z})X(w,\bar{w})$) suggests that X is is a primary of weight $(h,\bar{h}) = (0,0)$. However, this is <u>not</u> the case.

- To see this, note that in any QFT with an (unbroken) symm., correlation fcts. ($\equiv$ vac. exp. values) are covariant under that symmetry. [The most familiar example is Lorentz symm, e.g.

$$\langle U_\Lambda A^\mu(\Lambda x) U_\Lambda^+ U_\Lambda A^\nu(\Lambda y) U_\Lambda^+ \rangle = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \langle A^\rho(x) A^\sigma(y) \rangle$$

$$= \langle A^\mu(\Lambda x) A^\nu(\Lambda y) \rangle$$

trf. of operators $\qquad$ invariance of theory & vacuum]

Hence, in particular

$$\langle O_1'(z',\bar{z}')\, O_2'(w',\bar{w}') \rangle = \left(\frac{\partial z'}{\partial z}\right)^{-h_1} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}_1} \left(\frac{\partial w'}{\partial w}\right)^{-h_2} \left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{-\bar{h}_2} \cdot$$

$$\cdot \langle O_1(z,\bar{z})\, O_2(w,\bar{w}) \rangle$$

&

$$\langle O_1(z',\bar{z}')\, O_2(w',\bar{w}') \rangle$$

must coincide. For $h_1 = \bar{h}_1 = h_2 = \bar{h}_2 = 0$, and for $z' = \lambda z$; $w' = \lambda w$, we find that the correlator must be independent of the distance $|z-w|$, which is clearly <u>not</u> the case for $\ln|z-w|$. In this sense X is not a "good field" of the free-boson CFT.

- Our simplest primary is, instead, $\partial X(z)$. Following the same logic as above, we find:

$$\langle \partial X(z) \partial X(w) \rangle = \partial_z \partial_w \left(-\frac{\alpha'}{2} \ln|z-w|^2\right) = -\frac{\alpha'}{2} \cdot \frac{1}{(z-w)^2}$$

$$T(z)\partial X(w) = -\frac{1}{\alpha'} :\partial X(z) \partial X(z): \partial X(w) = \frac{\partial X(z)}{(z-w)^2} + \cdots$$

$$= \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \cdots \quad \Longrightarrow \quad \partial X \text{ is primary with weight } (h,\bar{h}) = (1,0).$$

9.3 The central charge

- From our previous discussion of trf. properties of corr. fcts., we would have expected $\partial X(z)\partial X(w) \sim \frac{1}{(z-w)^2}$.

This is also natural because ∂X has (naiv) (mass) dimension 1 [think of α' & hence X as dim.less and let z have dim. -1, as would be common in 4d QFT]. Then, corr. fct. $\sim 1/(z-w)^2$ is enforced by dim. analysis & holomorphicity.

- We immediately conclude (since $[T_{ab}] = d$ in general), that $[T(z)] = 2$ and thus

$$T(z)T(w) \sim \frac{1}{(z-w)^4} + \text{less singular}$$

- We define the central charges $c, \tilde{c}$ by

$$\boxed{T(z)T(w) = \frac{c/2}{(z-w)^4} + \text{less sing.}}$$ (and analogously for $\overline{T}(\bar{z})$..)

- We now work this out explicitly for the free boson:

$$T(z)T(w) = \frac{1}{\alpha'^2} :\partial X(z)\partial X(z)::\partial X(w)\partial X(w): = \cdots \quad (\to \text{problems})$$

$$\cdots = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \underset{\substack{\uparrow\\ \text{non-singular}}}{\cdots} \quad (c = 1!)$$

- We see from the non-zero first term, that $T(z)$ is not a primary.
- We still can use the coeff. of the second term to define the weight:

$$(h, \bar{h}) = (2, 0) \quad [\text{the "0" comes from } \overline{T}(\bar{z})T(w)].$$

- We apply our residue formula for generic δO: $\delta T(w) = -\text{Res}[\epsilon(z)T(z)T(w)]$ to derive:

$$\delta T(w) = -\epsilon(w)\partial T(w) - 2\epsilon'(w)T(w) - \frac{c}{12}\epsilon'''(w)$$

- The finite version of this is

$$T'(z') = \left(\frac{\partial z'}{\partial z}\right)^2 \left(T(z) - \frac{c}{12} S(z', z)\right)$$

"Schwarzian derivative": $S(z', z) \equiv \left(\frac{\partial^3 z'}{\partial z^3}\right)\left(\frac{\partial z'}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 z'}{\partial z}\right)^2 \left(\frac{\partial z'}{\partial z}\right)^{-2}$

- With $z = e^{-iw}$, this implies

$$T_{cylinder}(w) = -z^2 T_{plane}(z) + \frac{c}{24},$$

and with $\langle T_{plane} \rangle = 0$, it follows that

$$\langle H \rangle = \frac{1}{4\pi\alpha'} \int d\sigma \langle T_{00} \rangle = -\frac{1}{4\pi\alpha'} \int d\sigma \left(T_{ww} + \overline{T}_{\bar{w}\bar{w}}\right) = -\frac{\pi(c+\tilde{c})}{24} \cdot \left(\frac{1}{4\pi\alpha'}\right)$$

(This agrees with our Casimir energy calculation of Sect. 3 after $\alpha' = \frac{1}{2}$, which was our convention in Sect. 3:

$$\langle H \rangle = E_{Casimir} = -\frac{c+\tilde{c}}{2} \cdot \frac{1}{24} = -\frac{1}{24}.)$$

- Yet another connection to our previous discussion follows from observing that $[L_m, L_n]$ can be viewed as a conf. tf. (generated by L_m) acting on (the Fourier coefficient L_n of) T_{ab}. Hence, in

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m),$$

c is indeed identical to "D" (cf. Sect. 3), which we already had called "central charge" in that section.

- Finally, it is in principle straightforward to also calculate c_g, arising from $T^g T^g$ – OPE (where T^g follows from S^{ghost}):

$\Rightarrow \quad c_g = -26$; together with $c_X = \tilde{c}_X = 1$

$\Rightarrow \quad c = 0$ for 26 bosons + ghosts $\Rightarrow$ "true" CFT in $D=26$

9.4 Central charge vs. Weyl anomaly

- Let's now covariantly couple our CFT (e.g. that of X, b, c) to gravity and put into a curved background.

- While $T_a{}^a = 0$ classically and $\langle T_a{}^a \rangle = 0$ in QM can be achieved by normal ordering, a non-zero value due to local curvature can now not be avoided:

$$\langle T_a{}^a \rangle = \underset{\substack{\uparrow \\ \text{some coeff.}}}{b} \cdot \underset{\substack{\uparrow \\ \text{Ricci scalar}}}{\mathcal{R}} \left(+ \sim \mathcal{R}^2/\Lambda^2 \text{ etc., going to zero as } \Lambda \to \infty \right)$$

(leading term on dimensional grounds: $[\mathcal{R}] = 2$)

- Now go to $z, \bar{z}$-coordinates: $T_{\bar{z}z} = \frac{b}{2} h_{\bar{z}z} \mathcal{R}$ (Check by $\cdot h^{\bar{z}z}$)

- Take covariant derivative: $D^{\bar{z}} T_{\bar{z}z} = \frac{b}{2} D^{\bar{z}} h_{\bar{z}z} \mathcal{R}$

- Use $D^{\bar{z}} T_{\bar{z}z} + D^z T_{zz} = 0$ & covariant constance of h:

$$-D^z T_{zz} = \frac{b}{2} \partial_z \mathcal{R}$$

- Perform a Weyl-trf. with infinitesimal ω:

<u>r.h.-side:</u> $\delta_\omega \mathcal{R} = -2 D^2 \omega \Rightarrow \frac{b}{2} \partial_z \mathcal{R} = -4b\, \partial_z^2 \partial_{\bar{z}} \omega$

<u>l.h.-side:</u> Weyl = conf. trf. "minus" diffeomorphism

$$\delta_\omega T = -\frac{c}{12} \partial_z^3 \epsilon \underbrace{- 2(\partial_z \epsilon) T - \epsilon \partial_z T \text{ “}-\text{” diffeom. effect}}_{\text{find zero after cancellation of conf. \& diff. effects (as expected classically)}}$$

- Use $2\omega = \partial_z \epsilon + \partial_{\bar{z}} \bar{\epsilon}$ to find: $\delta_\omega T = -\frac{c}{6}\partial_z^2\omega \Rightarrow \delta_\omega \partial^{\bar z} T = -\frac{c}{6}\partial^{\bar z}\partial_z^2\omega$

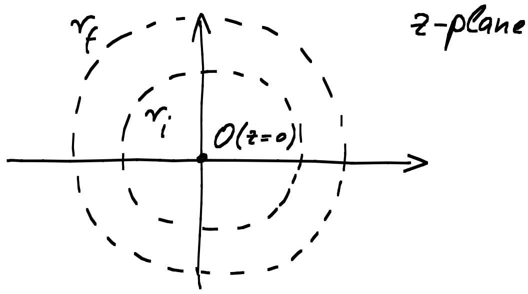
$$= -\frac{c}{3}\partial_{\bar z}\partial_z^2\omega$$

- Combine: $\frac{c}{3}\partial_{\bar z}\partial_z^2\omega = -4b\,\partial_z^2\partial_{\bar z}\omega \Rightarrow b = -\frac{c}{12}$

$$\boxed{T_a{}^a = -\frac{c}{12}\mathcal{R}}$$

$\Rightarrow$ Our full WS theory (with non-trivial background) is Weyl invariant if $c = 0$, i.e. for $D = 26$.

9.5 State - operator correspondence



- In analogy to the Schrödinger wave fct. of QM, a state in QFT can be characterized by a wave functional

$$\Psi : \varphi \longmapsto \Psi[\varphi] \in \mathbb{C}$$

(here $\varphi : \sigma \longmapsto \varphi(\sigma) \in \mathbb{R}$ is a field configuration at some fixed time τ)

- Thus, the state at "time" r_f evolving from a state Ψ_i at "time" r_i is given by

$$\Psi_f[\varphi_f, r_f] = \int \mathcal{D}\varphi_i \int\limits_{\substack{\varphi(\sigma, r_i) = \varphi_i(\sigma) \\ \varphi(\sigma, r_f) = \varphi_f(\sigma)}} \mathcal{D}\varphi\, e^{-S[\varphi]}\, \Psi_i[\varphi_i, r_i] \qquad (*)$$

- If we now start our evolution at $\tau = -\infty$, i.e. $r = 0$, i.e $z = 0$, we can characterize our initial state by an operator $O(z=0)$:

$$\Psi_i[\varphi_i, r_i] = \int\limits_{\varphi(\sigma, r_i) = \varphi_i} \mathcal{D}\varphi\, e^{-S[\varphi]}\, \underbrace{O(z=0)}_{\substack{\text{some expression in terms} \\ \text{of } \varphi(0),\, \partial\varphi(0),\, \bar\partial\varphi(0),\, \dots}}$$

- Thus, we have defined a map $\mathcal{O} \mapsto |\mathcal{O}(z=0)\rangle$ (the latter being defined by a Ψ_i at any $r_i > 0$).

- Inversely, consider a state defined by Ψ_i at any $r_i > 0$ (the r-dependence of Ψ_i is fixed by the dynamics, i.e. by S, as explained above). In particular, the corresponding Ψ_f at r_f is given by eq. (*).

- Now let's take $r_i \to 0$:

$$\Psi_f[\varphi_f, r_f] = \int_{\substack{\varphi(\sigma, r_f) = \varphi_f(\sigma) \\ (\text{any } \varphi \text{ at } z=0)}} \mathcal{D}\varphi \, e^{-S[\varphi]} \underbrace{\lim_{r_i \to 0} \Psi_i[\varphi(\sigma, r_i), r_i]}_{\substack{\text{some expression in } \varphi \text{ at } z=0 \\ \text{\& its derivatives} \\ \equiv \mathcal{O}(z=0)}}$$

- This completes our discussion by showing how to get $\mathcal{O}$ from the state (such that this $\mathcal{O}$ will give the state $|\mathcal{O}(z=0)\rangle$ we started from).

- We now understand that an amplitude with operator insertions based on the initial state $|\mathcal{O}(z=0)\rangle$ is given by

$$\int_{\cdots} \mathcal{D}\varphi \, e^{-S} \cdots \mathcal{O}(z=0)$$

final state & insertions

- Replacing $|\mathcal{O}(z=0)\rangle$ by $L_n |\mathcal{O}(z=0)\rangle$ gives

$$\int \mathcal{D}\varphi \, e^{-S} \cdots L_n \mathcal{O}(z=0) = \int \mathcal{D}\varphi \, e^{-S} \cdots \int \frac{dz}{2\pi i} z^{n+1} T(z) \mathcal{O}(z=0)$$

$$= \int \mathcal{D}\varphi \, e^{-S} \cdots \int \frac{dz}{2\pi i} z^{n+1} \left(\frac{h \, \mathcal{O}(z=0)}{z^2} + \frac{\partial \mathcal{O}(z=0)}{z} + \cdots \right)$$

This is non-singular if $\mathcal{O}$ is a primary.

- We see that: $L_{-1}|\mathcal{O}\rangle = |\partial\mathcal{O}\rangle$

$$L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle$$

$L_n|\mathcal{O}\rangle = 0$ for all $n>0$ iff $\mathcal{O}$ is a primary

- Thus, now appealing to the stringy origin of our CFT, we see that phys. states are primary states with $h=1$ (i.e. states corresponding to primary operators with weight 1).

  [For the BRST quantized string with ghosts, we need to appeal to the presence of a state with no ghost excitation in each equivalence class. The relevant L's are just the L^X.]

9.6 State-operator correspondence for the X-CFT

- Recall that $X(w,\bar w) = x + \alpha' p\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n e^{inw} + \tilde\alpha_n e^{in\bar w}\right)$

  (with $w = \sigma + i\tau$)

$$\partial_w X(w,\bar w) = -\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n e^{inw} \qquad \left(\text{with } \alpha_0 = i\sqrt{\tfrac{\alpha'}{2}}\,p\right)$$

$$\Big\downarrow \text{ tf. property of a primary}$$

$$\partial_z X(z,\bar z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \frac{\alpha_n}{z^{n+1}} \qquad (\text{with } z = e^{-iw})$$

$$\underset{\text{residue theorem}}{\Rightarrow} \quad \boxed{\alpha_n = i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}\, z^n\,\partial X(z)}$$

- Let's apply α_n $(n\geq 0)$ to $|1\rangle$:

$$\langle\cdot\cdot|\cdot\cdot\,\alpha_n|1\rangle = \int_{\cdots}\mathcal{D}X\cdot\cdot\, e^{-S}\, i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}\, z^n\,\partial X(z) = 0$$

since ∂X is holomorphic

$$\Rightarrow\ |1\rangle = |0, k=0\rangle$$

$$\Rightarrow\ \alpha_{-n}|0,k=0\rangle \mathrel{\hat{=}} i\sqrt{\frac{2}{\alpha'}}\oint\frac{dz}{2\pi i}\, z^{-n}\,\partial X(z)\cdot 1 = \cdots$$

$$\ldots = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i}\, z^{-n} \frac{1}{(n-1)!} z^{n-1} \partial^{n-1} \partial X(0) = i\sqrt{\frac{2}{\alpha'}} \cdot \frac{\partial^n X(0)}{(n-1)!}$$

- Thus, we now have all the operators corresponding to the string vacuum and the excited states, but it's so far only the vacuum with $k^\mu = 0$. We obviously need $k^\mu \neq 0$!

- The correct generalization is:

$$|0, k=0\rangle \;\hat{=}\; 1$$

$$\boxed{|0, k\rangle \;\hat{=}\; :e^{ik\cdot X(0)}:}$$

- The easiest way to see this is to recall that $e^{i\epsilon\cdot\hat{p}}$ shifts the $\hat{x}$-eigenvalue of a state. Analogously (since the Heisenberg algebra is basically symmetric in $\hat{x}, \hat{p}$), the operator $e^{ik\cdot\hat{x}}$ shifts the $\hat{p}$-eigenvalue of a state by k.

- Another check is that $:e^{ik\cdot X}:$ acquires a phase $e^{ik\cdot\epsilon}$ under a shift by ϵ^μ in X^μ.

- Finally, we can calculate the OPE with ∂X:

$$\partial X^\mu(z) :e^{ik\cdot X(w,\bar{w})}: = \ldots = -\frac{i\alpha' k^\mu}{2} \cdot \frac{1}{z-w} \cdot :e^{ik\cdot X(w,\bar{w})}:$$

From this the eigenvalue of α_0^μ follows.

- It is also crucial that (by explicit evaluation of the OPE with $T, \bar{T}$), $:e^{ik\cdot X}:$ is a primary with $h = \bar{h} = \frac{\alpha' k^2}{4}$.

see problems!

- The requirement of weight 1 (see above) now translates in the familiar mass shell condition.