Introduction to String Theory

Notes for the course of A. Hebecker in the winter term 2025/26

The course is strongly influenced by the standard textbooks [1–3]. The reader may also want to consult my old handwritten notes and further references available on the Web page [4]. I am indebted to Jan Louis, from whom I learned much of the subject, and to my former colleague Timo Weigand, whose set of lecture notes is an excellent resource [5], for many helpful discussions. Last but not least, I am very grateful to Daniele Berruti for his help in creating this LaTeX version of the notes.

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1 Introduction

With very minor exceptions, our world is perfectly described by the lagrangian

$$\mathcal{L} = \mathcal{L}_{grav} + \mathcal{L}_{SM} \ . \tag{1.1}$$

Here $\mathcal{L}_{SM}$ specifies a conventional QFT (scalars, fermions, gauge fields) and the most important exceptions are dark matter, baryogenesis and cosmological inflation.

The Einstein-Hilbert lagrangian

$$\mathcal{L}_{grav} = \frac{1}{2}\, M_P^2\, \mathcal{R}[g_{\mu\nu}] \tag{1.2}$$

may be viewed as specifying a classical field theory with the field being

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{1.3}$$

We take $\eta_{\mu\nu}$ to be the Minkowski metric, such that $h_{\mu\nu}$ may be viewed as describing fluctuations around this flat background. Without going much beyond what you learned in your QFT lectures, the field $h_{\mu\nu}$ may be quantized. The result is a slightly less conventional QFT, namely one with spin-2 particles, also known as gravitons. Symbolically, i.e. suppressing the index structure, one finds

$$\mathcal{L}_{grav} = \frac{1}{2}\, M_P^2 \left((\partial h)^2 + h(\partial h)^2 + h^2(\partial h)^2 + \cdots\right). \tag{1.4}$$

After the field redefinition $h_{\mu\nu} \to h_{\mu\nu}/M_P$, one has (still symbolically)

$$\mathcal{L}_{grav} = \frac{1}{2}(\partial h)^2 + \frac{1}{2M_P} h(\partial h)^2 + \frac{1}{2M_P^2} h^2(\partial h)^2 + \cdots \ . \tag{1.5}$$

We are now ready to discuss the key differences between $\mathcal{L}_{SM}$ and $\mathcal{L}_{grav}$: In the former, couplings are dimensionless, allowing for the crucial feature of renormalizability. Let us recall this in some detail, focusing on the QCD or strong-interaction part of the Standard Model:

$$\mathcal{L}_{SM} \supset \mathcal{L}_{QCD}\ , \qquad \text{with} \qquad \mathcal{L}_{QCD} = \frac{1}{2g^2}\,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}\,. \tag{1.6}$$

Here the key player is the gauge potential (the gluon field)

$$A_\mu \in Lie(SU(3))\,, \tag{1.7}$$

based on which we define

$$D_\mu = \partial_\mu + iA_\mu \qquad \text{and} \qquad F_{\mu\nu} = -i[D_\mu, D_\nu] \in Lie(SU(3))\,. \tag{1.8}$$

Working out the original lagrangian gives, as before symbolically,

$$\mathcal{L}_{QCD} = \frac{1}{2g^2}\left((\partial A)^2 + A^2(\partial A) + A^4\right), \tag{1.9}$$

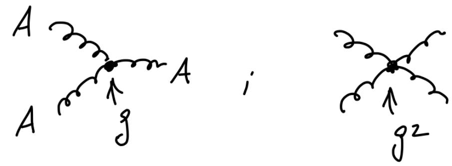

Figure 1: Feynman diagrams for QCD interactions.

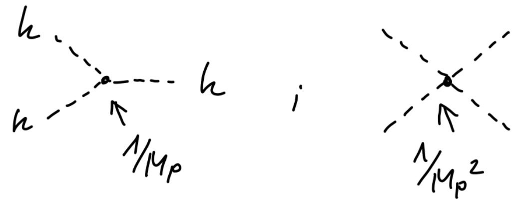

Figure 2: Feynman diagrams for perturbative quantum gravity.

or, after the redefinition $A_\mu \to gA_\mu$,

$$\mathcal{L}_{QCD} = \frac{1}{2}(\partial A)^2 + \frac{1}{2} g A^2 (\partial A) + \frac{1}{2} g^2 A^4 \ . \tag{1.10}$$

The Feynman diagrams associated with these interaction vertices are given in Fig. 1.

While we have analogous diagrams in gravity, see Fig. 2, the coupling there is dimensionful. We note in passing that one frequently defines the coupling constant to be $\kappa = 1/M_P$. Let us very briefly recall the standard QFT argument for why having a dimensionless coupling constant is crucial for renormalizability: When you calculate higher and higher loop corrections to some given process, you get more and more powers of g in your result. This, however, does not change the mass dimension of the rest of your result. Hence, superficially, the divergences that are potentially present have no reason to become worse at higher orders of perturbation theory.

For gravity, higher loop corrections correspond to higher powers of $\kappa = 1/M_P$, which in turn correspond to higher powers of Λ for dimensional reasons and hence worse and worse UV divergences. As a result, gravity works well as an "effective QFT" for $E << M_P$, but one faces total loss of control for $E \sim M_P$.

The origin of the UV divergences is in the point-like interactions, i.e. the integral

$$\int d^4x \, d^4y D^2(x-y) \tag{1.11}$$

diverges at $x - y \ \to \ 0$. This is illustrated in Fig. 3.

The key idea for improvement is to replace point particles by strings. As can be seen in Fig. 4, there is no "interaction point" or "vertex" on the r.h. side. There is only an "interaction region", the size of which is set by the typical size $1/M_s$ of the string. UV divergences are hence automatically cut off at the so-called string scale M_s.

As a basis for such "string-string scattering", we will need to quantize a single free string, cf. Fig. 5. After quantization, one finds a discrete set of vibrational modes with different masses

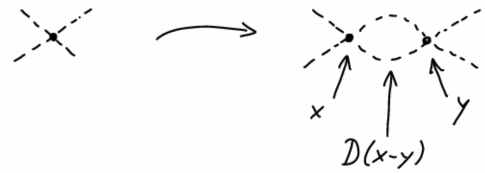

Figure 3: Origin of the UV divergences in QFT.

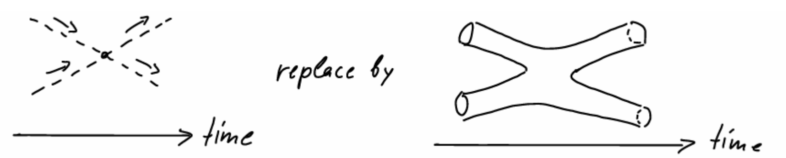

Figure 4: From particles to strings.

and spins. At low energies, only the lightest (as we will see, massless) such modes are relevant. They correspond to the particles (including a graviton) of the D-dimensional low-energy QFT – the EFT of string theory.

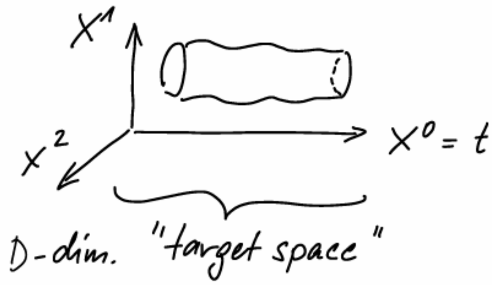

Figure 5: Free string in D-dimensional spacetime.

As we will see, the spacetime dimension D is not arbitrary but fixed by consistency requirements of the quantization process. Explicitly, we will find:

Bosonic string: $D = 26$ (unfortunately, the 26d QFT has an unstable vacuum)

Superstring: $D = 10$ (more or less unique 10d QFT – a 10d "supergravity theory").

To describe our world, we need to "compactify" 6 of the 10 dimensions. In other words, we consider solutions of the above 10d SUGRA of type $\mathcal{M}_{10} = \mathcal{R}^{1,3} \times \mathcal{M}_6$, with $\mathcal{M}_6$ a compact manifold, in general with certain VEVs of different fields of the SUGRA. Since there are potentially very many (even in the simplest settings $\sim 10^{500}$) such choices of $\mathcal{M}_6$ with appropriate VEVs, one speaks of the "string theory landscape", cf. Fig. 6. Here $V(\varphi)$ is the potential of the 4d EFT resulting from the compactification. The field φ may be thought of as parameterizing the space of different manifolds $\mathcal{M}_6$.

There are many open problems:

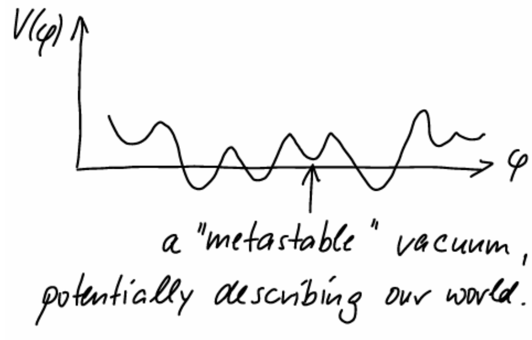

Figure 6: The string theory landscape.

- What does does the landscape look like at a detailed, quantitative level? To see how pressing this is, note that even the existence of metastable vacua with SUSY-breaking and positive cosmological constant - as in our world - is at the moment not established.
- How is the landscape populated in cosmology?
- How to treat time-dependent gravitational backgrounds, in particular singularities, in string theory?

Independent reasons for studying string theory arise at a more theoretical level:

- String Theory is an indispensable tool of modern research in QFT and GR. It provides an explicit UV completion respecting Poincaré symmetry and, if desired, supersymmetry or SUSY.
- Via AdS/CFT, which is deeply rooted in ST, it allows for the study of non-perturbative effects in both QFT and GR. Here 'AdS/CFT' stands for an equivalence (also known as a "duality") between GR in Anti-de-Sitter space and a specific conformal field theory (without gravity) in one dimension less.

2 Classical Bosonic String

2.1 Relativistic point particle

The relativistic point particle (cf. Fig. 7) is a very useful toy model, exhibiting some key aspects of the string. You should have already studied it in your course on special relativity. The embedding of worldline γ of the particle in $\mathcal{M}$ is specified by D functions $X^\mu(\tau)$, where τ parameterizes γ. The action, known from relativity, can be expressed in terms of these functions:

$$S = \text{“length of worldline”} = -m\int_\gamma ds = -m\int d\tau\,\sqrt{-\eta_{\mu\nu}\,\dot{X}^\mu\dot{X}^\nu}. \tag{2.1}$$

Here we used the metric

$$ds^2 = -\eta_{\mu\nu}\, dX^\mu dX^\nu, \qquad \text{such that} \qquad \mathcal{M} = \mathbb{R}^{1,\,D-1} \qquad \text{and} \qquad dX^\mu = \dot{X}^\mu d\tau\,, \tag{2.2}$$

with $\hbar = c = 1$.

Figure 7: Worldline of a particle in D-dimensional spacetime.

You should be able to check that:

- S is invariant under reparameterizations: $\tau \to \tau'(\tau)$.
- The EOMs read $\ddot{X}^\mu = 0$.
- The non-relativistic limit is $S = \int dt\,(\frac{1}{2}m\vec{v}^2 - m)$.

The action above is the point-particle analogue of the so-called "Nambu-Goto action" of the string, to be discussed very soon. We will hence write: $S = S_{\rm NG}$.

Both for the point particle and for the string, another useful action is the "Polyakov action" $S_{\rm P}$, obtained as follows: Recall that on a manifold with coordinates y^a one measures distances using a metric: $ds^2 = g_{ab}\, dy^a dy^b$. Treat γ as a 1d manifold, with metric $ds^2 = h_{\tau\tau}\, d\tau^2$. A general action on γ would then be

$$S = \int d\tau \sqrt{-h}\, \mathcal{L}(X^\mu, \dot{X}^\mu)\,. \tag{2.3}$$

The specific choice

$$S_{\rm P}[X, h] \equiv -\frac{m}{2} \int d\tau \sqrt{-h}\Big(h^{\tau\tau}\, \dot{X}^\mu \dot{X}_\mu + 1\Big) \tag{2.4}$$

is called the Polyakov action. Here $h \equiv \det h \equiv h_{\tau\tau}$, which is of course slightly tautological for a 1×1 matrix. Moreover, $h^{\tau\tau} = h^{-1}_{\tau\tau}$.

The EOM for the metric h follows, as usual, form the variational principle: $\delta S_{\rm P}/\delta h_{\tau\tau} = 0$. One can check that they explicitly read

$$h_{\tau\tau} = \dot{X}^\mu \dot{X}_\mu \equiv \dot{X}^2 \tag{2.5}$$

and that

$$S_{\rm P}[X,\, h = \dot{X}^2] = S_{\rm NG}\,, \tag{2.6}$$

Hence $S_{\rm P}$ and $S_{\rm NG}$ are classically equivalent. $S_{\rm P}$ is more convenient since it has no square root.

2.2 Bosonic string

Everything proceeds analogously: The embedding of the worldsheet Σ in target space $\mathcal{M}$ is specified by functions $X^\mu(\tau,\sigma)$, cf. Fig. 8. The action reads

$$S_{\rm NG} = -T \int_\Sigma df\,, \tag{2.7}$$

where T is the string tension (the analogue of the mass m) and $\int_\Sigma df$ is the area of Σ, measured with the target-space metric $\eta_{\mu\nu}$. It will be convenient to use a covariant coordinate notation also on Σ:

$$(\tau,\sigma) \equiv (\xi^0,\xi^1) \equiv \xi^a. \tag{2.8}$$

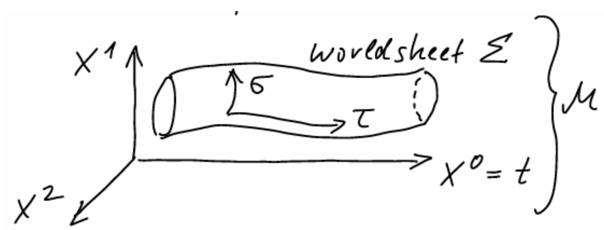

Figure 8: String worldsheet in D-dimensional spacetime.

An infinitesimal translation $d\xi$ on Σ induces an infinitesimal translation dX on M. It follows that

$$ds^2 = -\eta_{\mu\nu}\, dX^\mu dX^\nu = -\eta_{\mu\nu}\left(\frac{\partial X^\mu}{\partial \xi^a}d\xi^a\right)\left(\frac{\partial X^\nu}{\partial \xi^b}d\xi^b\right) = -G_{ab}\, d\xi^a d\xi^b\,. \tag{2.9}$$

We see that $G_{ab} = \partial_a X^\mu\, \partial_b X^\nu\, \eta_{\mu\nu}$ is the **induced metric** on Σ. Hence, we may write more explicitly

$$S_{\rm NG} = -T\int_\Sigma d^2\xi\, \sqrt{-G} \qquad \text{with} \qquad G = \det G_{ab}\,. \tag{2.10}$$

In almost complete analogy to the point-particle case, we introduce an independent worldsheet metric h_{ab} and define the Polyakov action:

$$S_{\rm P} = -\frac{T}{2}\int_\Sigma d^2\xi\, \sqrt{-h}\, h^{ab}\, \partial_a X^\mu\, \partial_b X^\nu\, \eta_{\mu\nu}. \tag{2.11}$$

A key difference is that no constant term is needed to achieve classical equivalence with $S_{\rm NG}$. We will show this in a moment. Note that $S_{\rm P}$ is a field theory action for D free real scalars in two dimensions.

A central object in the analysis of such theories is the energy-momentum tensor,

$$T_{ab} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\rm P}}{\delta h^{ab}}. \tag{2.12}$$

This definition differs from the standard GR convention by a "stringy" normalization factor -2π. To calculate T_{ab} explicitly, we first rewrite the Polyakov action as

$$S_P = -\frac{T}{2}\int d^2\xi\, \sqrt{-h}\, h^{ab} G_{ab}\,. \tag{2.13}$$

We first observe that

$$\delta(h^{ab}G_{ab}) = \delta h^{ab}G_{ab}. \tag{2.14}$$

Next, the variation of $\sqrt{-h}$ is given by

$$\delta\sqrt{-h} = -\frac{1}{2\sqrt{-h}}\delta(\det h) = -\frac{1}{2\sqrt{-h}}(\det h)\,\mathrm{tr}(h^{-1}\delta h)\,. \tag{2.15}$$

The last equality represents a famous identity for the variation of the determinant of a generic matrix. We leave the proof to the reader.

With the rewriting

$$\mathrm{tr}(h^{-1}\delta h) = -\mathrm{tr}(h\,\delta h^{-1}) = -h_{ab}\delta h^{ab} \tag{2.16}$$

we obtain

$$T_{ab} = \frac{4\pi}{\sqrt{-h}}\left(-\frac{T}{2}\right)\left(\sqrt{-h}\,G_{ab} + h^{cd}G_{cd}\left(-\frac{h}{2\sqrt{-h}}\right)(-h_{ab})\right)\,. \tag{2.17}$$

This simplifies to

$$T_{ab} = -2\pi T\left(G_{ab} - \frac{1}{2}h_{ab}(G_{cd}h^{cd})\right) = -\frac{1}{\alpha'}\left(G_{ab} - \frac{1}{2}h_{ab}(G_{cd}h^{cd})\right)\,, \tag{2.18}$$

where in the last step we introduced the "Regge slope" α'. This name goes back to the early days of string theory, which was originally invented as a model for hadronic physics. The key idea is illustrated in Fig. 9. In this model, the tension was observed to be a proportionality factor between mass m^2 and angular momentum or spin J. The linear trajectory following from this observation is in reasonably good agrement with data, as illustrated in Fig. 10.

Figure 9: Naive model of a hadron as a rotating open string, i.e. a string with two open ends.

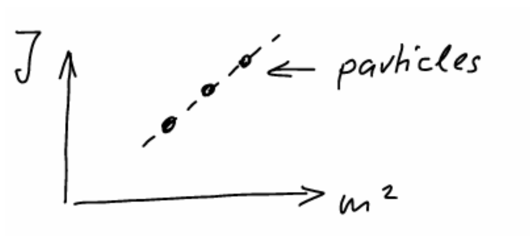

Figure 10: Hadrons shown as points in a plane spanned by mass squared and angular momentum.

Finally, it is clear that the EOMs for the metric h are simply $T_{ab} = 0$. They are solved by $h_{ab} = c\,G_{ab}$ for any function c. Indeed, evaluating the r.h. side of (2.18) by using the relation

$$\frac{1}{2}cG_{ab}\left(c^{-1}G^{cd}G_{cd}\right) = G_{ab}\,, \tag{2.19}$$

one concludes that $T_{ab} = 0$. Moreover, substituting our solution h_{ab} back in the Polyakov action, we find

$$S_{\rm P}[X, h_{ab} = cG_{ab}] = -\frac{T}{2}\int d^2\xi\,\sqrt{-c^2\,G}\,c^{-1}G^{ab}G_{ab} = -T\int d^2\xi\,\sqrt{-G} = S_{\rm NG}[X]\,. \qquad (2.20)$$

2.3 Equations of Motion and Symmetries

Recall that

$$S_P = -\frac{T}{2}\int d^2\xi\sqrt{-h}(\partial X)^2, \quad \text{with} \quad (\partial X)^2 = h^{ab}(\partial_a X^\mu)(\partial_b X^\nu)\eta_{\mu\nu}\,, \qquad (2.21)$$

where h_{ab} is the worldsheet metric and $\eta_{\mu\nu}$ is the metric on the "field space" of our 2d QFT. The symmetries of this action are:

1. Diffeomorphisms:
$$\xi^\alpha \;\to\; \xi'^a = \xi'^a(\xi^0, \xi^1)\,. \qquad (2.22)$$

2. D-dimensional Poincaré invariance:
$$X^\mu \;\to\; X'^\mu = \Lambda^\mu{}_\nu x^\nu + V^\mu \qquad \text{with} \qquad \Lambda \in SO(1, D-1)\,. \qquad (2.23)$$
We note that this may be viewed as an internal global symmetry of our 2d QFT.

3. Weyl-rescaling invariance:
$$h_{ab}(\xi) \;\to\; h'_{ab}(\xi) = \varphi(\xi)\,h_{ab}(\xi)\,. \qquad (2.24)$$
The fact that such a rescaling factor $\varphi(\xi)$ drops out of the action is a very important special feature of the worldsheet dimension, $d = 2$.

For completeness, we record the EOMs of h and X:

$$h_{ab} \;: \qquad T_{ab} = 0\,, \qquad (2.25)$$
$$X^\mu \;: \qquad \Box X^\mu = 0\,, \qquad (2.26)$$

where $\Box \equiv D_a\partial^a$. The former was derived above, the latter is a standard QFT result.

Two comments are in order: First, as in GR, diffeomorphism invariance implies $D_a T^{ab} = 0$ even "off-shell", i.e. before the equations of motion of h set T_{ab} to zero. Second, the relation $T^a_a = 0$ also holds as an identity, i.e. without using the equations of motion. We leave it to the reader to derive this latter fact from the symmetries of $S_{\rm P}$.

2.4 Choosing a gauge

Both diffeomorphisms and Weyl rescalings do not affect the embedding of Σ in $\mathcal{M}$. Therefore, we declare them to be **gauge symmetries**, i.e. they relate different descriptions of the same physics. A key claim is that using diffeomorphisms and Weyl rescalings, we can locally ensure

$$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.27}$$

This is called the "flat gauge".

A naive argument for this claim goes as follows: A diffeomorphism together with a Weyl rescaling,

$$\xi^\alpha \to \xi'^a = \xi'^a(\xi^0, \xi^1) \qquad \text{and} \qquad h_{ab} \to h_{ab}\, \varphi(\xi^0, \xi^1)\,, \tag{2.28}$$

are specified by 3 arbitrary functions: ξ'^0, ξ'^1 and φ. Together, they define how the metric changes. Since the metric h_{ab} itself contains only 3 arbitrary functions, we generally have enough freedom to bring h_{ab} to any desired form.

A more precise argument can be given by using the Ricci scalar $\mathcal{R}[h]$ of the worldsheet with metric h_{ab}: First, a straightforward calculation (see e.g. [6]) shows that

$$h'_{ab} = e^{2\omega} h_{ab} \qquad \Rightarrow \qquad \mathcal{R}[h'] = e^{-2\omega}\left(\mathcal{R}[h] - 2D^2\omega\right). \tag{2.29}$$

Given some metric h, we can now solve the PDE $D^2\omega = \frac{1}{2}\mathcal{R}[h]$ for ω. This is a simple wave equation with a source. On a cylinder, with some cut through the cylinder as a Cauchy surface, this will always have a solution. Having found ω, we rescale $h_{ab} \to h'_{ab} = e^{2\omega} h_{ab}$. Now we have a gauge-equivalent metric h' with $\mathcal{R}[h'] = 0$. Specifically in $d = 2$, we have

$$R_{abcd} = \frac{1}{2}\big(h_{ab}h_{cd} - h_{ad}h_{bc}\big)\mathcal{R}\,. \tag{2.30}$$

Thus, our new metric has vanishing Riemann tensor. By standard GR knowledge, this implies that the metric is in fact flat. In other words: One can choose coordinates such that $h_{ab} = \mathrm{diag}(-1, 1)$. This is called a **flat gauge**. A more general gauge choice which is also frequently used are the **conformal gauges**. They are defined by the requirement that h_{ab} is flat up to a rescaling. In other words, $h_{ab} = e^{2\omega}\,\mathrm{diag}(-1, 1)$.

Note that we will later consider the Euclidean version of our 2d theory. Then worldsheets Σ other than the cylinder (or strip, in the case of the so-called open string) will become relevant. The existence and uniqueness of a flat gauge choice will become highly non-trivial and important. We refer to [3], Sec. 2.3 and 6.2, for more powerful mathematical methods used in this context.

2.5 Solutions

We use flat gauge and light-cone coordinates: $\sigma^\pm = \tau \pm \sigma$. Then the metric takes the form

$$ds^2 \equiv -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-, \tag{2.31}$$

i.e.

$$h_{++} = h_{--} = 0, \qquad h_{+-} = h_{-+} = -\frac{1}{2}, \qquad h^{+-} = h^{-+} = -2. \tag{2.32}$$

The d'Alembertian becomes

$$\Box = h^{ab}\partial_a\partial_b = 2h^{+-}\,\partial_+\partial_- = -4\,\partial_+\partial_- \qquad \text{with} \qquad \partial_\pm \equiv \frac{\partial}{\partial\sigma^\pm}\,. \tag{2.33}$$

Thus, the equations of motion take the form

$$\partial_+\partial_- X^\mu = 0\,. \tag{2.34}$$

It is immediately clear that any solution can be written as

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)\,, \tag{2.35}$$

where the index L/R stands for left/right-moving wave. These names are explained by the parametrization of the cylinder as

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+l)\,, \tag{2.36}$$

see also Fig. 11. By diffeomorphism and Weyl invariance, we can choose any desired value for l and still keep the flat metric.

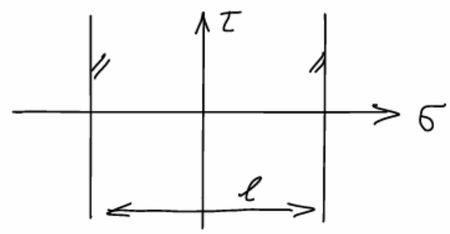

Figure 11: Parametrization of the cylinder.

Since X^μ is periodic in σ, it follows that $\partial_+ X^\mu_L$ and $\partial_- X^\mu_R$ are both periodic in σ. Hence they can be written as a sum over the basis of functions $\exp(-2\pi i n\sigma^\pm)\,,\;\; n\in\mathbb{Z}$. The fields X^μ_L and X^μ_R follow by integration and, as a result, also contain a linear term. Thus, we have shown that the general solution takes the form

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{l}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}^\mu_n\, e^{-2\pi i n\sigma^+/l}\,, \tag{2.37}$$

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{l}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\, e^{-2\pi i n\sigma^-/l}\,. \tag{2.38}$$

Here the apparently complicated naming of the coefficients is conventional. We will see the usefulness of these conventions below. Since only the sum of $X^\mu_{L/R}$ is physically meaningful, we can change both arbitrarily by adding/subtracting a constant. This choice has been used to make the constant term $x^\mu/2$ the same on both sides.

The coefficient $(\pi\alpha'/l)\,p^\mu$ of the linear term must be the same in $X^\mu_{R/L}$ for periodicity of X^μ in σ to hold. Moreover, the fact that X^μ is real implies reality of x^μ, p^μ and the relations $(\alpha^\mu_n)^* = \alpha^\mu_{-n}$, $(\tilde\alpha^\mu_n)^* = \tilde\alpha^\mu_{-n}$. Finally, we note that the sum of $X^\mu_{L/R}$ takes the form

$$X^\mu = x^\mu + \frac{2\pi\alpha'}{l}\,p^\mu\,\tau + \dots\,, \tag{2.39}$$

which is to be interpreted as the superposition of linear motion (shown explicitly) and fluctuations (replaced by the ellipsis).

As noted, the naming of the coefficients in the **oscillator expansion** just introduced is conventional and we follow [3]. Clearly, different conventions exist. For example, [1] (cf. also my old notes) choose $l=\pi$ and, in addition, define a so-called "string length" $l_s = \sqrt{2\alpha'}$. Thus,

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{l_s^2}{2}p^\mu\,\sigma^+ + \frac{il_s}{2}\sum_{n\neq 0}\frac{1}{n}\,\tilde\alpha^\mu_n\,e^{-2in\sigma^+}\,, \tag{2.40}$$

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{l_s^2}{2}p^\mu\,\sigma^- + \frac{il_s}{2}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\,e^{-2in\sigma^-}\,. \tag{2.41}$$

The quantities l and l_s are conceptually different and should not be confused. Also, please be aware that the string length and the string scale $M_s \sim 1/l_s$ differ by constant factors between the definitions used by various authors. By contrast, the definition of α' is, the best of my knowledge, universally accepted.

3 Old covariant quantization – basic setup

Before quantizing, we need to go from the Lagrangian to the Hamiltonian formulation. For systems with gauge invariance, which corresponds to systems with constraints, this is highly non-trivial. Useful references include [7–10]. We will not be able to present the very interesting and important story of the **quantization of systems with constraints** but will only use some basic elements to get the 'stringy' results of interest for us. Nevertheless, before doing so, it is worthwhile to try and understand some of the basic issues using our simple toy model: the relativistic point particle.

3.1 Towards the Hamiltonian treatment of the relativistic particle

We recall that

$$S_{\text{NG}} = -m\int d\tau\,\sqrt{-\dot X^\mu \dot X_\mu}, \qquad X^\mu = X^\mu(\tau), \quad \dot X^2 = \dot X^\nu \dot X_\nu\,. \tag{3.1}$$

The canonical momenta are then given by[1]

$$p_\mu = \frac{\partial\mathcal{L}}{\partial\dot X^\mu} = m\frac{\dot X^\mu}{\sqrt{-\dot X^2}}\,. \tag{3.2}$$

[1]It would at this point look more consistent to use a capital 'P_μ', to match 'X^μ'. We refrain from doing so because the analogous momentum variable in the string context, cf. last chapter, is traditionally denoted by a small letter, i.e. as p_μ.

Using this object, define the function

$$\Phi = p^2 + m^2 \tag{3.3}$$

and work it out:

$$p^2 + m^2 = m^2 \frac{\dot{X}^2}{\left(\sqrt{-\dot{X}^2}\right)^2} + m^2 = 0\,. \tag{3.4}$$

This identity is true even before imposing the equations of motion. This is called a **primary constraint**. Its presence implies that it is impossible to express X^μ in terms of p^μ. We must hence restrict ourselves to the submanifold of phase space defined by the constraint.

The canonical Hamiltonian is

$$\mathcal{H} = \dot{X}^\mu \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} - \mathcal{L} = -m\sqrt{-\dot{X}^2} - \left(-m\sqrt{-\dot{X}^2}\right) = 0\,. \tag{3.5}$$

The fact that it vanishes identically is due to time-diffeomorphism invariance, which is also responsible for the primary constraint found above. But we will not try to understand this in general. In the presence of constraints, the Hamiltonian may be modified as follows:

$$\mathcal{H} \;\to\; \mathcal{H} + \sum_k c_k\,\phi_k\,. \tag{3.6}$$

In other words, we are allowed to add a linear combination of constraints since these vanish on the constrained phase space. In our case, there is just one constraint and, with the renaming $c_1 \to N$, we have

$$\mathcal{H} = \frac{N}{2m}\,(p^2 + m^2)\,. \tag{3.7}$$

The Hamilton equation for the evolution of X^μ in Poisson bracket form reads

$$\dot{X}^\mu = \{X^\mu, \mathcal{H}\} = \frac{N}{m}\,p^\mu\,. \tag{3.8}$$

This implies

$$N^2 = -\dot{X}^2\,. \tag{3.9}$$

We see that by choosing our free function N, also known as Lagrange multiplier, we are in fact choosing the time variable. Choosing $N = 1$ corresponds to the choice of eigentime: $d\tau^2 = -dX^2$. In summary, we lose one of the p^μ to the constraint and one of the X^μ to gauge invariance, i.e. to the arbitrary choice of the function $N(\tau)$. Thus, our phase space has only 6 of the naively expected 8 dimensions.

Alternatively, one may also start with the Polyakov-type action

$$S_{\mathrm{P}} = -\frac{m}{2}\int d\tau\,\sqrt{-h}\,(h^{-1}\dot{X}^2 + 1)\,, \tag{3.10}$$

which implies

$$p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-h}}\,. \tag{3.11}$$

There is an obvious primary constraint in that the canonical momentum of h vanishes, simply because $\dot{h}$ does not appear in the action. There is also a so-called secondary constraint arising from the dynamics, specifically from the equations of motion for h:

$$-\frac{1}{2\sqrt{-h}}(h^{-1}\dot{X}^2+1)-\sqrt{-h}(h^{-2}\dot{X}^2)=0 \qquad \Rightarrow \qquad \frac{\dot{X}^2}{(-h)}+1=0. \tag{3.12}$$

Using (3.11), we recognize our old constraint $p^2+m^2=0$. We now naively have a 10d phase space, but we lose two degrees of freedom to the two constraints and two more to gauge invariance: One because of the free choice of time and the other because of the free choice of h. In the end, a 6d phase space is left, as before. One can show that gauge transformations are in fact generated by the constraints, in the Hamiltonian sense.

It is easy to derive the equations of motion for X^μ:

$$\partial_\tau\left(\frac{\dot{X}^\mu}{\sqrt{-h}}\right)=0\,. \tag{3.13}$$

Using the h-equation of motion, i.e. the secondary constraint, it follows that

$$\partial_\tau\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}\right)=0\,. \tag{3.14}$$

This is the expected result that trajectories are given by straight lines.

If we were to quantize the system, it would obviously be wrong to focus only on X^μ and p^μ, although these are of course canonical variables. However, they are not independent, which can be most clearly seen from the fact that not all straight lines are solutions. Indeed, requiring $\ddot{X}^\mu=0$ is not sufficient. We **must** in addition demand $p^2+m^2=0$. This constraint is crucial both in the classical Hamiltonian and later in the quantum formalism.

To summarize, we may write

$$\frac{\delta S}{\delta h}\sim \text{“}T_{ab}\text{”}=0 \qquad \Longleftrightarrow \qquad p^2+m^2=0\,. \tag{3.15}$$

This must be enforced in the canonical quantization. As before, the origin lies in the gauge-invariace of our system under time-diffeomorphisms (see [3, 11] for more details). As a final comment, we note one may impose the constraints first and quantize later or the other way round. The former is known as **reduced phase space quantization** for obvious reasons, the latter is frequently referred to as **Dirac quantization**.

3.2 Quantizing the closed string

Let us keep the discussion of constraints above in mind, but proceed naively at first. We use the Polyakov action in flat gauge:

$$S=-\frac{T}{2}\int d^2\xi\,\partial^\alpha X_\mu\,\partial_\alpha X^\mu\,. \tag{3.16}$$

This is just a 2d QFT with D scalars X^μ. The canonical momentum reads

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T\dot{X}_\mu \tag{3.17}$$

and the Hamiltonian is obtained according to

$$\mathcal{H} = \int_0^l d\sigma\, (\dot{X}^\mu \Pi_\mu - \mathcal{L}) = \frac{T}{2}(\dot{X}^2 + X'^2) = T((\partial_+ X)^2 + (\partial_- X)^2)\,. \tag{3.18}$$

We calculate the Poisson-Bracket between momenta and coordinates and then postulate the corresponding commutator relation, introducing the familiar facto of 'i' on the r.h. side:

$$[\Pi_\mu(\tau,\sigma),\, X^\nu(\tau,\sigma')] = -i\,\delta(\sigma-\sigma')\,\delta_\mu{}^\nu, \qquad [X,X] = [\Pi,\Pi] = 0. \tag{3.19}$$

We recall the oscillator expansion given earlier, where now

$$(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m}, \qquad (\tilde{\alpha}^\mu_m)^\dagger = \tilde{\alpha}^\mu_{-m}, \tag{3.20}$$

by the hermiticity of the X^μ. It is straightforward to work out that

$$[\alpha^\mu_m,\, \alpha^\nu_n] = m\,\delta_{m+n}\,\eta^{\mu\nu}, \qquad [\tilde{\alpha}^\mu_m,\, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n}\,\eta^{\mu\nu}, \qquad [p^\mu,\, x^\nu] = -i\,\eta^{\mu\nu}\,, \tag{3.21}$$

where we used the shorthand notation $\delta_k \equiv \delta_{k,0}$. We also note that the integral of the 'momentum density' Π^μ belonging to the field X^μ reads:

$$P^\mu = \int_0^l d\sigma\,\Pi^\mu = T\int_0^l d\sigma\,\dot{X}^\mu = \frac{1}{2\pi\alpha'}\int_0^l d\sigma\,\frac{\partial}{\partial\tau}\left(\frac{2\pi\alpha'}{l}\,p^\mu\,\tau\right) = p^\mu. \tag{3.22}$$

This proves that the coefficient p^μ in our expansion is really the physical 4-momentum of our string.

The reader should check the commutation relations between the oscillator coefficients stated above. For example, one may simply integrate (3.19) using the relation

$$\int_0^l d\sigma \int_0^l d\sigma'\, e^{2\pi i m\sigma/l}\, e^{2\pi i n\sigma'/l}\,\delta(\sigma-\sigma') = l\,\delta_{m+n}\,. \tag{3.23}$$

Note that the terms with x^μ, p^ν do not appear in conventional QFT since in this case $l=\infty$. As a result the quantity x^μ, i.e. the average field value, is non-dynamical.

3.3 Quantizing the open string

Everything, from classical analysis to quantization, can be repeated for the open string: We simply replace the geometry of a circle by that of an interval, cf. Fig. 12 . As before, we parametrize the worldsheet by $\sigma \in (0,\ell)$ and τ, see Fig. 13.

The Polyakov action takes the same form as before:

$$S = -\frac{T}{2}\int d^2\xi\, (\partial^\alpha X^\mu)\,(\partial_\alpha X_\mu)\,. \tag{3.24}$$

Figure 12: Transition from closed to open string.

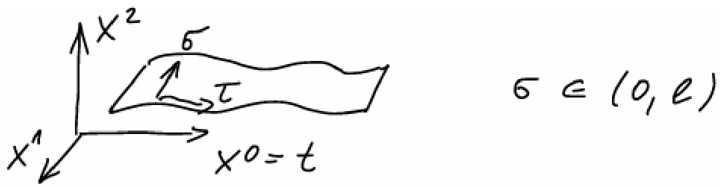

Figure 13: Open string in D-dimensional spacetime.

But we need to rethink the derivation of the equations of motion. We have

$$\delta S = -T \int d^2\xi \; (\partial^\alpha X^\mu)\,(\partial_\alpha \delta X_\mu) = T \int d^2\xi \, (\partial^2 X^\mu)\, \delta X_\mu - T \int_{-\infty}^{+\infty} d\tau \, \big(\partial^\sigma X^\mu\big)\delta X_\mu \Bigg|_{\sigma=0}^{\sigma=\ell} \,, \tag{3.25}$$

where the first term on the r.h. side defines the equations of motion. The second term is a boundary term which vanishes for:

1. Neumann boundary conditions: $\partial_\sigma X^\mu\big|_{\sigma=0,\ell} = 0\,,$
2. Dirichlet boundary conditions: $\delta X^\mu\big|_{\sigma=0,\ell} = 0\,.$

The physical interpretation of this is as follows:

In the **Neumann:** case, the shift symmetry in X^μ is preserved and hence momentum conservation respected. Momentum cannot flow off the string at the endpoints, so the endpoints move freely in target space.

In the **Dirichlet** case, $X^\mu(\tau, 0)$ and $X^\mu(\tau, \ell)$ are fixed, so the endpoints are attached to what is known as a **D-brane**.

In general, some of the X^μ obey Neumann boundary conditions (move freely), while the others obey Dirichlet boundary conditions (are fixed). This corresponds to the string ending on a D-brane which takes the form of a hyperplane in target space. This hyperplane is defined by fixing the values of those X^μ for which we have Dirichlet boundary conditions, cf. fig. 14.

The following naming convention is widely accepted: A D-brane with p spatial dimensions is called a ‘Dp-brane’. The string ending on it then has $(p+1)$ Neumann and $(D-p-1)$ Dirichlet boundary conditions. Our figure shows a D1-brane in $D = 3$ target space. Strings can end on different branes and hence have different boundary conditions at each end, cf. Fig. 15.

One can derive mode expansions for NN, DD, ND, and DN boundary conditions. We only

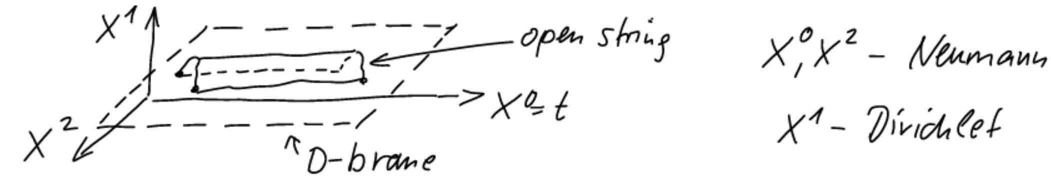

Figure 14: Example of string ending on a D-brane.

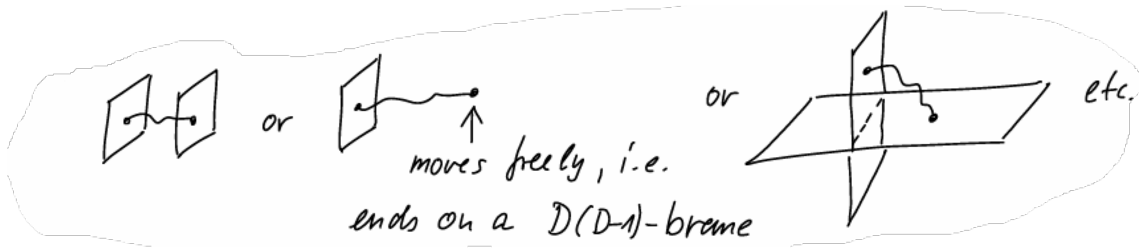

Figure 15: Different boundary conditions for open strings.

give the NN case:

$$X^\mu(\tau,\sigma) = x^\mu + \frac{2\pi\alpha'}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-i\pi n\tau/l}\cos\left(\frac{\pi n\sigma}{l}\right). \tag{3.26}$$

The key difference is that, to respect the Neumann boundary conditions, the σ-dependence is given by appropriate cosine modes. The prefactors of the oscillator terms have been correspondingly adjusted. A decomposition into left and right-moving parts is not useful since these are not independent. This is intuitively clear since e.g. a left moving wave turns into a right moving upon reflection at the boundary.

For DD, ND, and DN the p^μ-term is absent. This is clear since, in these cases, the center of mass of the open string as a whole can not propagate in the brane-transverse directions. The x^μ-term is fixed by the D-brane position(s). The reader should try to work out the corresponding mode decompositions, see also [3]. We finally note that a situation with Neumann boundary conditions for all X^μ may be thought of as the string ending on a $D9$-brane, which is obviously spacetime-filling.

A very useful way to think about the open string is to start with the closed string on a doubled interval, $\sigma \in (-l, l)$, with (symbolically)

$$X^\mu = \ ... + \sim \sum_{n\neq 0}\frac{i}{n}\left(\tilde{\alpha}_n^\mu e^{-2i\pi n\sigma^+/l} + \alpha_n^\mu e^{-2i\pi n\sigma^-/l}\right) \tag{3.27}$$

and demand symmetry under $\sigma \mapsto -\sigma$ ("mod out a $\mathbb{Z}_2$"). This enforces Neumann boundary conditions at the 'fixed points' of the involution: $\sigma = 0$ and $\sigma = 1$. It also enforces $\alpha_n^\mu = \tilde{\alpha}_n^\mu$. Thus:

$$X^\mu(\tau,\sigma) = ...+ \sim \sum_{n\neq 0}\frac{i}{n}\alpha_n^\mu e^{-2i\pi n\tau/l}\cos\left(\frac{2\pi n\sigma}{l}\right). \tag{3.28}$$

We see again that the left-and right-moving excitations are not independent, as argued above from the reflecting boundary conditions. In summary, we have

$$[p^\mu, x^\nu] = -i\eta^{\mu\nu} \tag{3.29}$$

for the NN case and

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\,\eta^{\mu\nu} \tag{3.30}$$

for all open strings. In other words, we have 'half' of the closed string modes.

3.4 Constructing the Fock space

Our operator algebra contains the α^μ_m, in the closed string case, the $\tilde{\alpha}^\mu_m$. They describe the oscillators and obey the hermiticity relation $\alpha^{\mu\,\dagger}_m = \alpha^\mu_{-m}$. It also contains the momentum and position operators $\hat{p}^\mu$, $\hat{x}^\mu$, which are both hermitian.[2]

We assume that a representation on a Hilbert $\mathcal{H}$ exists. One can diagonalize $\hat{p}^\mu$ or $\hat{x}^\mu$. We choose $\hat{p}^\mu$, as appears natural in the particle physics context. Thus,

$$\mathcal{H} = \bigoplus_p \mathcal{H}(p)\,, \tag{3.31}$$

where $\mathcal{H}(p)$ is the subspace on which $\hat{p} = (\hat{p}_0, \ldots, \hat{p}_{D-1})$ is diagonal with eigenvalues $p = (p_0, \ldots, p_{D-1})$. Now we focus on one of the eigenspaces $\mathcal{H}(p)$, such that $\hat{p}$ is replaced by the real vector p. We build a representation of the α s using

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\,\eta^{\mu\nu}\,. \tag{3.32}$$

We may also restrict this to $m > 0$ and write

$$[\alpha^\mu_m, \alpha^{\nu\dagger}_n] = m\delta_{m,n}\,\eta^{\mu\nu}\,, \tag{3.33}$$

making it obvious that this is just a set of harmonic oscillators with non-standard normalization.

As usual, one defines the vacuum $|0,p\rangle$ by

$$\alpha^\mu_m|0,p\rangle = 0 \quad \forall m > 0,\, \forall\mu,\, \forall p\,. \tag{3.34}$$

Finally, we define

$$\mathcal{H}(p) = \mathrm{span}\big\{\, \alpha^\mu_m\alpha^\nu_n \cdots |0,p\rangle\,\big\}\,. \tag{3.35}$$

Here any number of α^μ_m, α^ν_n, $\cdots$ with $m, n, \ldots < 0$ may appear. This applies to any p.

Crucially, this holds in the above form for the open string. For the closed string, the discussion is completely analogous, but one uses in addition the same relations for the $\tilde{\alpha}^\mu_m$ and, obviously, also allows them to appear on the r.h. side of (3.35).

An immediate problem is the following: While $[\alpha^i_m, \alpha^{i\dagger}_m] = +m$ for $i, j = 1, 2\cdots(D-1)$, as it should be, one also has $[\alpha^0_m, \alpha^{0\dagger}_m] = -m$. This is not suitable for the standard oscillator

[2]We suppress the hat symbols on the α s but keep them on p and x for reasons to become clear momentarily.

interpretation. One could think of 'switching the roles' of (for $m > 0$) annihilator α_m^0 and creator $\alpha_m^{0\dagger}$, but that would destroy the Lorentz-transformation properties of our representation. In fact, the wrong-sign commutator is disastrous for the physics interpretation since

$$[a, a^\dagger] = -1 \quad \Rightarrow \quad |a^\dagger|0\rangle|^2 = \langle 0|aa^\dagger|0\rangle = -\langle 0|0\rangle \,. \tag{3.36}$$

This implies that $a^\dagger|0\rangle$ is a negative-norm state or 'ghost' (not to be confused with the acceptable and useful Faddeev–Popov ghosts).

One inspiration for how to proceed comes from QED. Recall that there the field A_μ gave rise to anninhilation/creation operators a_μ, $a_\mu^\dagger$, with $a_0, a_0^\dagger$ having a wrong-sign commutator. Within Gupta-Bleuler quantization, the resolution was to impose the (covariant or Lorentz) gauge-fixing condition on all physical states:

$$(\partial_\mu A^\mu)\Big|_{\text{annihilator part}} |\text{phys}\rangle = 0\,. \tag{3.37}$$

Note however that this is not perfectly analogous to our situation. Indeed, the object we gauge fix is the worldsheet metric, which is eventually completely eliminated as a physical degree of freedom. Thus, there is no analogue of the Lorentz gauge condition above in our case.

Another inspiration comes from the relativistic particle. There, we saw that the constraint $p^2 + m^2 = 0$ **had** to be imposed in the Hamiltonian framework. This constraint came from the equation of motion $\delta S/\delta h = 0$. It has remained crucial even after the worldline metric h disappeared by gauge fixing.

Here we can proceed in exact analogy. We have

$$\frac{\delta S}{\delta h^{ab}} = 0 \qquad \Rightarrow \qquad T_{ab} = 0. \tag{3.38}$$

All we need to do is to implement this at the quantum level. This will be implemented following the idea from QED mentioned earlier.

It will be convenient to use light-cone coordinates σ^+ and σ^-. We recall that $T_a{}^a = 0$ holds as an identity. This translates into $T_{-+} = T_{+-} = 0$. Hence, we only need to impose $T_{++} = 0$ and $T_{--} = 0$. From the general expression

$$T_{ab} = -\frac{1}{\alpha'}\left(G_{ab} - \frac{1}{2}h_{ab}(G_{cd}h^{cd})\right) \tag{3.39}$$

we obtain, using flat gauge and light-cone coordinates,

$$T_{--} = -\frac{1}{\alpha'}(\partial_- X)\cdot(\partial_- X) \qquad \text{and} \qquad T_{++} = -\frac{1}{\alpha'}(\partial_+ X)\cdot(\partial_+ X)\,. \tag{3.40}$$

Recalling that $\partial_+ X_R = 0$ and $\partial_- X_L = 0$ this turns into

$$T_{++} = -\frac{1}{\alpha'}(\partial_+ X_L)\cdot(\partial_+ X_L) \qquad \text{and} \qquad T_{--} = -\frac{1}{\alpha'}(\partial_- X_R)\cdot(\partial_- X_R)\,. \tag{3.41}$$

It will be convenient to work not with T_{++} and T_{--} themselves but with their Fourier modes at $\tau = 0$:

$$L_m \equiv -\frac{l}{4\pi^2}\int_0^\ell d\sigma\, e^{2\pi i m\sigma/l}\, T_{--} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\alpha_n \tag{3.42}$$

$$\tilde{L}_m \equiv -\frac{l}{4\pi^2}\int_0^\ell d\sigma\, e^{2\pi i m\sigma/l}\, T_{++} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\tilde{\alpha}_n\,. \tag{3.43}$$

Here we have introduced the convenient notation

$$p^\mu\sqrt{\alpha'/2} \equiv \alpha_0^\mu \equiv \tilde{\alpha}_0^\mu \qquad\qquad \text{(closed string)}\,. \tag{3.44}$$

The reader should carefully check that the relations (3.42),(3.43) hold. Conceptually, it is clear what happens: T is a product of two sums, each going over all modes. This results in a double sum, with the oscillation of each term governed by the sum of the two relevant wave numbers. The Fourier coefficient with index m picks out those contributions where the sum of the two wave numbers is precisely m. One is left with a single sum.

For the open string, the left and right movers and hence T_{++} and T_{--} are not independent. As described in more detail in [3], one starts with a cylinder of circumference $2l$ and mods out by $\mathbb{Z}_2$ (the so-called doubling trick sketched earlier). The Fourier modes are then $\exp(i\pi m\sigma/l)$ and one may define

$$L_m \equiv -\frac{l}{2\pi^2}\int_0^\ell d\sigma\, \left(e^{\pi i m\sigma/l}\, T_{++} + e^{-\pi i m\sigma/l}\, T_{--}\right) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\alpha_n\,. \tag{3.45}$$

These L_m contain all the relevant information. To allow for the compact notation on the r.h. side, the α_0 s have to be defined differently from the closed case:

$$p^\mu\sqrt{2\alpha'} \equiv \alpha_0^\mu \equiv \tilde{\alpha}_0^\mu \qquad\qquad \text{(open string)}\,. \tag{3.46}$$

It is straightforward to check:

$$L_m^\dagger = L_{-m}; \quad \tilde{L}_m^\dagger = \tilde{L}_{-m}; \quad L_0 + \tilde{L}_0 = \frac{l}{2\pi}H\,. \tag{3.47}$$

Here H is the hamiltonian. For the open string, it is instead given by

$$L_0 = \frac{l}{\pi}H\,. \tag{3.48}$$

As a crucial point, we note that L_0 has an ordering ambiguity since α_m and α_{-m} stand next to each other. Thus, the expressions above should be understood as holding only at the classical level. At the quantum level, we have to define L_0 (or re-define) L_0 taking the ordering ambiguity into account. We do so as follows:

$$\boxed{L_0 \equiv \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n \quad \text{(normal ordered).}} \tag{3.49}$$

With this definition (and some work, for which we refer to the tutorials) one can show that the Ls obey the so-called **Virasoro algebra**:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\,\delta_{m+n} \qquad \text{with} \qquad A(m) = \frac{D}{12}(m^3 - m)\,. \tag{3.50}$$

Here $(m-n)L_{m+n}$ is known as the classical part and $A(m)$ as the anomaly.

A comment is in order: At the Hamiltonian level, before quantization, we would have found the **Witt algebra**:

$$\{L_m, L_n\} = (m-n)L_{m+n}\,. \tag{3.51}$$

This reflects simply the classical reparametrization invariance of the S^1. The reason is that T_{ab} (being the current belonging to P_a) generates local translations, i.e. diffeomorphism. To see the Witt algebra arise explicitly, consider the space of functions on a circle, $f(\theta) = f(\theta + 2\pi)$. Local translations, $f(\theta) \to f(\theta + a(\theta))$, are generated by operators

$$D[a] = a(\theta)\,\partial_\theta. \tag{3.52}$$

Consider the basis $D_n = ie^{in\theta}\partial_\theta$ and derive its Lie algebra:

$$[D_m, D_n] = [\,ie^{im\theta}\partial_\theta,\ ie^{in\theta}\partial_\theta] = i(m-n)e^{i(m+n)\theta}\partial_\theta = (m-n)D_{m+n}\,. \tag{3.53}$$

This is indeed the Witt algebra

We now return to the quantum theory. We want to impose $T_{ab} = 0$, which is the same as $L_m = 0$, on physical states. But this is too strong since,

$$0 = \langle \text{phys}|[L_m, L_{-m}]|\text{phys}\rangle = A(m)\langle \text{phys}|\text{phys}\rangle \neq 0\,. \tag{3.54}$$

In other words, we find a contradiction. For our purposes it suffices to demand

$$(L_m + a\,\delta_m)\,|\text{phys}\rangle = 0\,, \quad \forall m \geq 0\,. \tag{3.55}$$

The constant a is a so far unknown and accounts for the ordering ambiguity in L_0 which we have not yet tried to resolve. This is similar to what one does in the context of Gupta-Bleuler quantization of QED, where only the annihilation part of $\partial_\mu A^\mu$ is used to constrain physical states.

We note that condition (3.55) is sufficient to ensure that the expectation value of *all* L_m and hence of T_{ab} vanishes in a physical state:

$$\langle \text{phys}|L_m|\text{phys}\rangle = 0, \quad \forall m\,. \tag{3.56}$$

Indeed, take $m > 0$ and observe:

$$\langle \text{phys}|L_{-m}|\text{phys}\rangle = \overline{\langle \text{phys}|L_m|\text{phys}\rangle} = 0\,, \quad \text{as desired.} \tag{3.57}$$

4 Old covariant quantization - explicit construction of states

We have a Fock space $\mathcal{H}$ that is built on the direct sum of all $|0,p\rangle$. All creation operators α_m and $\tilde{\alpha}_m$ with $m<0$ can act on each of these vacua. The Physical Subspace is the set of all $|\psi\rangle$ in $\mathcal{H}$ with

$$(L_m + a\,\delta_m)|\psi\rangle = 0 \qquad \text{and} \qquad (\tilde{L}_m + a\,\delta_m)|\psi\rangle = 0\,, \tag{4.1}$$

for every $m \geq 0$.

4.1 Open string

Vacuum: For any p, we have a vacuum state $|0,p\rangle$ with $\hat{p}|0,p\rangle = p|0,p\rangle$. Let $m\geq 0$ and demand

$$0 = L_m|0,p\rangle = \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n|0,p\rangle\,. \tag{4.2}$$

If $m>0$, this is trivially satisfied in this case since either $n>0$ or $(m-n)>0$. We see that only the L_0-constraint is relevant:

$$0 = (L_0 + a)|0,p\rangle = \left(\alpha'\hat{p}^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n + a\right)|0,p\rangle = \left(\alpha'\hat{p}^2 + \hat{N} + a\right)|0,p\rangle\,. \tag{4.3}$$

where

$$\hat{N} \equiv \sum_{n>0}\alpha_{-n}\cdot\alpha_n \tag{4.4}$$

is the **level number operator**. This constraint is called the **mass-shell condition**. Since in our case $\hat{N}$ annihilates the state, we find

$$p^2 = -\frac{a}{\alpha'} \qquad \text{or} \qquad m^2 = -p^2 = \frac{a}{\alpha'}\,. \tag{4.5}$$

Thus, we found one scalar particle with mass squared equal to a/α'.

First excited level: A general state built by acting with a single 'level-one' creation operator can be written as $\zeta_\mu\,\alpha^\mu_{-1}|0,p\rangle$. It is characterized by the polarization vector ζ_μ. Now the mass-shell condition reads

$$0 = (L_0 + a)\;\zeta_\mu\,\alpha^\mu_{-1}|0,p\rangle = \left(\alpha'p^2 + \alpha^\nu_{-1}\alpha_{1\nu} + a\right)\zeta_\mu\,\alpha^\mu_{-1}|0,p\rangle\,. \tag{4.6}$$

Here we have dropped all terms except the first one from the infinite sum contained in L_0. Only this first term can contribute non-trivially in the case at hand.

Using the oscillator commutation relations gives

$$0 = (\alpha'p^2 + 1 + a)\;\zeta\cdot\alpha_{-1}\,|0,p\rangle\,, \tag{4.7}$$

where '$\cdot$' abbreviates the index contraction. Finally, one obtains

$$m^2 = \frac{1+a}{\alpha'} \,. \tag{4.8}$$

In addition to the mass-shell condition, now the L_1- condition also becomes non-trivial:

$$0 = L_1 \; \zeta \cdot \alpha_{-1} \, |0,p\rangle = \left(\frac{1}{2} \sum_n \alpha_{1-n} \cdot \alpha_n \right) \zeta \cdot \alpha_{-1} \, |0,p\rangle = 0 \,. \tag{4.9}$$

We see that,

- for $n \leq -1$, α_{1-n} always annihilates the state.
- for $n \geq 2$, α_n always annihilates the state.

Thus, we may restrict the sum to $n \in \{0,1\}$, which gives

$$0 = (\alpha_1\alpha_0 + \alpha_0\alpha_1)\, \zeta\cdot\alpha_{-1}|0,p\rangle \quad \Rightarrow \quad \zeta\cdot\alpha_0|0,p\rangle = 0 \quad \Rightarrow \quad \zeta\cdot p|0,p\rangle = 0 \quad \Rightarrow \quad \zeta\cdot p = 0 \,. \tag{4.10}$$

This means that the polarization vector must be transversal. Let us also calculate the norm of the state. We find

$$\langle 0,p|(\zeta \cdot \alpha_{-1})^\dagger \, (\zeta \cdot \alpha_{-1}) \, |0,p\rangle = \langle 0,p|0,p\rangle \; \zeta^2, \tag{4.11}$$

where $\zeta^2 \equiv \zeta_\mu \zeta^\mu$. As a result of this analysis, we can distinguish three cases for level-one states:

a) $a < -1 \;\Rightarrow\; m^2 < 0 \;\Rightarrow\; p$ space-like. In this case, there exists a time-like ζ with $\zeta \cdot p = 0$. This implies the presence of negative-norm states. The case is excluded.

b) $a = -1 \;\Rightarrow\; m^2 = 0 \;\Rightarrow\; p$ light-like. In this case, there are $(D-1)$ vectors ζ with $p \cdot \zeta = 0$:

   - one longitudinal, i.e. $\zeta \parallel p$. This is a zero-norm state.
   - $(D-2)$ transverse, for which $\zeta^2 > 0$. They have positive norm.

   This corresponds precisely to the physical states found in Gupta-Bleuler quantization of QED. We hence expect a $U(1)$ gauge theory to emerge. This is the case is of 'Critical String Theory'.

c) $a > -1 \;\Rightarrow\; m^2 > 0 \;\Rightarrow\; p$ time-like. In this case, there are $(D-1)$ space-like vectors ζ. We expect them to correspond to $(D-1)$ polarizations of a massive vector field. At the present level of analysis, this is OK. It will, however, turn out to be problematic at the interacting and higher-loop level. This case is known as 'Non-critical String Theory'. Consistency appears to require sacrificing target-space Poincaré invariance. Applicability of this case to real-world physics still unclear.

Second excited level: The general state is

$$|\psi\rangle = (\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1} + \epsilon_{\mu}\alpha^{\mu}_{-2})|0, p\rangle\,. \tag{4.12}$$

The mass-shell constraint reads

$$(L_0 + a)|\psi\rangle = 0\,, \qquad \text{implying} \qquad m^2 = \frac{2+a}{\alpha'}\,. \tag{4.13}$$

In addition, we now need to consider the L_1 and L_2 constraints. Using those, one has to work out allowed polarizations, masses, and norms of the corresponding physical states. We will not go into this in any detail since we will soon study approaches where the results we would find follow more easily. These are light-cone & BRST quantization – see below.

For now, we only present an argument for $D = 26$ based on a simple example state (more details can be found in [1]). We base our discussion on a state of the form

$$|\phi\rangle = \{c_1\,\alpha_{-1}\cdot\alpha_{-1} + c_2\,p\cdot\alpha_{-2} + c_3(p\cdot\alpha_{-1})^2\}|0, p\rangle. \tag{4.14}$$

Focus on the case $a = -1$, which we identified as particularly promising above, and impose the appropriate constraints:

$$(L_0 - 1)|\phi\rangle = 0; \quad L_1|\phi\rangle = 0, ; \quad L_2|\phi\rangle = 0\,. \tag{4.15}$$

From these, one may derive constraints on c_1, c_2, c_3. They allow on to express c_2, c_3 through c_1. Finally, calculating the norm of our state ϕ gives

$$\langle\phi|\phi\rangle = \frac{2|c_1|^2}{25}(D-1)(26-D)\,. \tag{4.16}$$

It follows that we need $D \leq 26$ for consistency. Moreover, we see that $D = 26$ is special because, like for level one, we obtain zero-norm states signaling a large gauge symmetry. In fact, a detailed analysis shows that there are in total many such states. We thus learn that, in some sense, $D = 26$ "belongs to" $a = -1$. In other words, the critical string mentioned before really has $\boldsymbol{a = -1}$ **and** $\boldsymbol{D = 26}$.

With more work, one can show that:

a) $a < -1$, excluded due to the presence of ghosts.

b) $a = -1$, $D = 26$ is a very special case governed by a large gauge symmetry. This makes the theory solvable and consistent including all levels, interactions, loops. This is the 'Critical String'. As we will see later, the key point here is that, in this case, Weyl symmetry straightforwardly survives quantization.

c) $a > -1$, $D < 26$ may be consistent with caveats. It is known as the 'Non-critical string'.

4.2 Formal summary of old covariant quantization

We start with a Fock space $\mathcal{H}$ and define subspace $\mathcal{H}_{\rm phys} \subset \mathcal{H}$ of 'physical states' by

$$(L_m + a\,\delta_m)|\text{phys}\rangle = 0 \quad (m \geq 0). \tag{4.17}$$

In this context, states $|\psi\rangle$ with $\langle\psi|\text{phys}\rangle = 0$ for all physical states are called 'spurious'. States which are both spurious and physical are called 'null'. We then have

$$\mathcal{H} \supset \mathcal{H}_{\rm phys} \supset \mathcal{H}_{\rm null}\,. \tag{4.18}$$

The possibility of adding states from $\mathcal{H}_{\rm null}$ to any physical states corresponds to the so-called 'residual gauge symmetry'. In the case of the critical string, this subspace is very large.

Finally, the proper Hilbert space is defined as

$$\underset{\sim}{\mathcal{H}} \equiv \mathcal{H}_{\rm phys}/\mathcal{H}_{\rm null}\,. \tag{4.19}$$

Here, all states are physical, in the sense of being allowed, and no states distinct from zero have vanishing norm. This is, of course, required by the definition of a Hilbert space.

We recall that, in our conventions, $m^2 = -p^2$ which allows us to write the mass-shell condition in the **critical case** and for the **open string** as:

$$m^2 = \frac{1}{\alpha'}(N-1)\,, \qquad\qquad \text{with} \qquad\qquad \hat{N} = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n\,. \tag{4.20}$$

The eigenvalue of N is called the **level**. The physical states at the different levels are:

- $N = 0$: tachyon;
- $N = 1$: massless vector;
- $N \geq 2$: excited string states.

Let us briefly comment on the name tachyon: In special relativity, more precisely for the special relativistic wave equation, one can show that the group velocity is $\overline{v} = \overline{p}/E$. With $m^2 = E^2 - \overline{p}^2 = E^2(1-\overline{v}^2)$, it follows that negative m^2 implies $|\overline{v}| > 1$, i.e. a faster-than light state appears to be present. However, this is not what really happens. A scalar with negative m^2 simply means that what we called the vacuum is, from the target-space prespective, a maximum of the potential. Due to quantum fluctuations, this state is short-lived – one eventually rolls down. This is called tachyon-condensation. Where precisely one ends up is in general unknown. Thus, our perturbative analysis based on this vacuum is not trustworthy except for very short time scales or as a toy model. This will only resolved when we turn to the superstring.

4.3 Closed string

We discuss only the critical case. As noted above, a better justification of this restriction will follow soon. For now, we simply focus on $D = 26$, $a = -1$.

Compared to the open string, we have twice as many oscillators and constraints: $L_m \to L_m,\ \tilde{L}_m$. It is useful to re-arrange the L_0-constraints

$$(L_0 + a)|\text{phys}\rangle = 0\ , \qquad (\tilde{L}_0 + a)|\text{phys}\rangle = 0 \tag{4.21}$$

as

$$(L_0 - \tilde{L}_0)|\text{phys}\rangle = 0\ , \qquad (L_0 + \tilde{L}_0 + 2a)|\text{phys}\rangle = 0. \tag{4.22}$$

Recalling that the relation between α_0 and p changes for closed strings, we also have

$$L_0 = N + \frac{\alpha_0^2}{2} = N + \frac{\alpha'}{4}p^2 \qquad \text{and} \qquad \tilde{L}_0 = \tilde{N} + \frac{\alpha_0^2}{2} = \tilde{N} + \frac{\alpha'}{4}p^2\,. \tag{4.23}$$

In this language, the constraints may be written as

(I) Level matching:

$$(N - \tilde{N})|\text{phys}\rangle = 0 \tag{4.24}$$

(II) Mass-shell condition:

$$\left(N + \tilde{N} + 2a + \frac{\alpha'}{2}p^2\right)|\text{phys}\rangle = 0\,. \tag{4.25}$$

As in the open case, we can now analyze each level:

Vacuum: $N = \tilde{N} = 0$ implying $m^2 = -p^2 = 4a/\alpha' = -4/\alpha'$. This state is known as the tachyon.

First excited level: $N = \tilde{N} = 1$ implies $m^2 = -p^2 = (4 + 4a)/\alpha' = 0$. Now the L_1 and $\tilde{L}_1$ constraints are also non-trivial. We write the general state at level 1 as

$$\xi_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0, p\rangle \tag{4.26}$$

and find

$$\xi_{\mu\nu}p^{\mu} = 0, \quad \xi_{\mu\nu}p^{\nu} = 0. \tag{4.27}$$

Formally, these are $2D$ constraints on the polarization tensor ξ. But we see that, given the first D constraints, a certain linear combination of the second set of constraints is already satisfied:

$$0 = (\xi_{\mu\nu}p^{\mu})p^{\nu} = p^{\mu}(\xi_{\mu\nu}p^{\nu})\,. \tag{4.28}$$

Thus, only $2D - 1$ constraints are independent and we are left with

$$D^2 - (2D - 1) \quad \text{physical states at level 1.} \tag{4.29}$$

It is easy to show that

$$\langle\text{phys}|\text{phys}\rangle \sim +\xi_{\mu\nu}\xi^{\mu\nu}. \tag{4.30}$$

As a result, we can immediately write down two sets of $(D-1)$ null states:

$$\xi^{(1)}_{\mu\nu} = \alpha_\mu p_\nu \quad \text{with} \quad \alpha \cdot p = 0 \qquad \text{and} \qquad \xi^{(2)}_{\mu\nu} = p_\mu \beta_\nu \quad \text{with} \quad \beta \cdot p = 0\,. \tag{4.31}$$

The linear space of null states has dimension $2(D-1)-1$ since the state with $\xi_{\mu\nu} \sim p_\mu p_\nu$ belongs to both sets. The null states correspond to gauge freedom, i.e.

$$\xi_{\mu\nu} \to \xi_{\mu\nu} + \xi^{(1)}_{\mu\nu} + \xi^{(2)}_{\mu\nu} \tag{4.32}$$

is a gauge transformation. The expected Hilbert space dimension is then

$$\dim \underset{\sim}{\mathcal{H}}_{N=1} \;=\; D^2 - (2D-1) - (2D-3) = (D-2)^2\,. \tag{4.33}$$

To see explicitly that, in this space, states are really physical with positive norm, choose

$$p = (1,1,0,\ldots,0). \tag{4.34}$$

Then one may define the physical or 'transverse' ξs to take the form of a $(D-2)\times(D-2)$ matrix:

$$\begin{pmatrix} 0_{2\times 2} & 0 \\ 0 & \xi_{t\,(D-2)\times(D-2)} \end{pmatrix}\,. \tag{4.35}$$

The matrix ξ_t transforms under the **little group** $SO(D-2)$, i.e., the subgroup of $SO(1,D-1)$ leaving p invariant. Under this subgroup, ξ_t is a rank-2 tensor. We may decompose the corresponding rank-2 tensor representations of $SO(D-2)$ of dimension $(D-2)^2$ into its irreducible components. The corresponding dimensions add up as follows:

$$(D-2)^2 = \binom{D-2}{2} + 1 + \left\{ \binom{D-2}{2} + (D-2) - 1 \right\} \tag{4.36}$$

Here the first term corresponds to antisymmetric tensors, giving rise to an antisymmetric tensor field $B_{\mu\nu}$. The second term is the trace, giving rise to a scalar field known as the dilaton ϕ. Finally, the rest is the symmetric traceless part of ξ_t, corresponding to the graviton $G_{\mu\nu}$.

The field $B_{\mu\nu}$ may be throught of as an analogue of the gauge potential A_μ with its field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, just with more indices. Here, the field strength is

$$H_{\mu\nu\rho} = 3\,\partial_{[\mu}B_{\nu\rho]}\,, \tag{4.37}$$

entering the D-dimensional lagrangian in the familiar way: $\mathcal{L}_D \supset\sim H_{\mu\nu\rho}H^{\mu\nu\rho}$.

Second excited level: We have $N=2$ and hence

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} + 2a) = \frac{4}{\alpha'}(N+a) = \frac{4}{\alpha'} \tag{4.38}$$

We find a large number of massive states which we will not discuss for the moment. The number of states grows very quickly as one goes to higher and hence even more massive levels.

Summary (critical) closed string: We emphasize the key points which differ from the open string discussed in greater detail above:

- There appears a level matching constraint: $N = \tilde{N}$;
- The mass-shell condition now reads: $m^2 = \dfrac{4}{\alpha'}(N-1)$.

At level 0 we find a tachyon. At level 1 we have the massless fields $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. Starting at level 2, there are massive string excitations.

5 Light-cone quantization – basic setup

5.1 Light-cone gauge

The flat gauge we used so far leaves some 'residual gauge freedom' unfixed. This freedom is related to the null states we have encountered. We now want to make this explicit and fix the gauge further. To do so, recall the action of general diffeomorphisms on coordinates and metric:

$$\xi^a \to \xi'^a = \xi^a(\xi)\,, \qquad\qquad h_{ab} \to h'_{ab}(\xi') = h_{cd}(\xi)\frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b}\,. \tag{5.1}$$

Considering the infinitesimal version of the action on coordinates,

$$\xi'^a \to \xi^a + \epsilon^a(\xi)\,, \tag{5.2}$$

we may write

$$h'_{ab}(\xi') = h'_{ab}(\xi) + \epsilon^c \partial_c h_{ab}(\xi)\,, \tag{5.3}$$

where we suppress terms of higher order in ϵ. Using this together with (5.1) gives

$$h'_{ab}(\xi) = h_{ab}(\xi) - \epsilon^c \partial_c h_{ab} - (\partial_a \epsilon^c h_{cb} + \partial_b \epsilon^c h_{ac}) = h_{ab} - (D_a \epsilon_b + D_b \epsilon_a)\,. \tag{5.4}$$

In the last step, we used that the metric is covariantly constant.

Let us now take the flat metric as our starting point. We then have

$$\eta_{ab} \to \eta_{ab} - (\partial_a \epsilon_b + \partial_b \epsilon_a)\,. \tag{5.5}$$

In addition, we may consider infinitesimal Weyl rescalings:

$$\eta_{ab} \to \eta_{ab}(1 + 2\omega)\,. \tag{5.6}$$

Such a Weyl rescaling compensates the diffeomorphism if

$$2\,\omega\,\gamma_{ab} = \partial_a \epsilon_b + \partial_b \epsilon_a\,. \tag{5.7}$$

It should now be clear that this specific combination of Weyl rescaling and diffeomorphism represents our residual gauge freedom. The above condition represents 3 independent equations, which in light-cone coordinates read

$$2\,\omega\,\eta_{+-} = \partial_+\epsilon_- + \partial_-\epsilon_+\,, \qquad \partial_+\epsilon_+ = 0\,, \qquad \partial_-\epsilon_- = 0\,. \tag{5.8}$$

The last two equations may as well be written as

$$\partial_+\epsilon^- = 0\,, \qquad \partial_-\epsilon^+ = 0\,. \tag{5.9}$$

This simply means that $\epsilon^- = \epsilon(\sigma^-)$ and $\epsilon^+ = \epsilon(\sigma^+)$. The first equation in (5.8) can then always be implemented by choosing an appropriate function ω.

We see that the group of residual gauge transformations is $(\mathrm{Diff}(S^1))^2$. We may recall that $\mathrm{Diff}(S^1)$ is generated by the Witt algebra classically and by the Virasoro algebra in the quantum theory. Not surprisingly, these algebras are precisely the algebras of our constraints. The latter correspond to the statement $T_{ab} = 0$ in flat gauge. Thus, our constraints simply said that physical states must be invariant under residual gauge freedom.

With this understanding, we clearly see the logical alternative to Old Covariant Quantization: We may fix the residual gauge freedom classically. Then we will need fewer constraints when we quantize. The reader may recall that this corresponds to going from Dirac quantization to reduced phase space quantization.

The gauge condition we want to use is roughly $\tau \sim X^+ + \text{const}$. In other words, we want to measure time on the worldsheet using the the light-cone target-space coordinate function X^+. By definition, there are then no X^+-oscillators.

To implement this explicitly, define

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1)\,. \tag{5.10}$$

Note that we use different normalization of the light-cone coordinates in target space compared to the worldsheet.

We will implement our proposed gauge fixing on solutions.[3] Consider a general solution X^+ of the equation of motion

$$\partial_+\partial_- X^+(\sigma^+,\sigma^-) = 0\,. \tag{5.11}$$

We want to ensure that X^+ coincides with our time variable $\tau = (\sigma^+ + \sigma^-)/2$ up to a freely chose proportionality factor. It is convenient to choose this factor such that the linear center-of-mass motion we found earlier corresponds to our evolution of X^+ in light-cone gauge. To see that this can be done, we first note that (5.11) is equivalent to

$$\partial_+\partial_-(X^+ - \frac{2\pi\alpha'}{l}p^\mu\tau) = 0\,, \tag{5.12}$$

with the factor in front of τ chosen as explained above. It follows that there exist $f(\sigma^+)$ and $g(\sigma^-)$ such that

$$X^+ - \frac{2\pi\alpha'}{l}p^+\tau \;=\; f(\sigma^+) + g(\sigma^-)\,. \tag{5.13}$$

[3]It can also be done off-shell, but this is not so easy to demonstrate and not essential for us. The reader may consult [11] for this issue.

We may define functions $\sigma'^+(\sigma^+)$ and $\sigma'^-(\sigma^-)$ such that

$$\frac{2\pi\alpha'}{l}p^+\,\frac{\sigma^+}{2} - f(\sigma^+) = \frac{2\pi\alpha'}{l}p^+\,\frac{\sigma'^+}{2}\,, \tag{5.14}$$

$$\frac{2\pi\alpha'}{l}p^+\,\frac{\sigma^-}{2} - g(\sigma^-) = \frac{2\pi\alpha'}{l}p^+\,\frac{\sigma'^-}{2}\,. \tag{5.15}$$

Using the residual gauge freedom, we may replace the world sheet light-cone coordinates with these functions. This corresponds to 'gauging away' f and g in (5.13), at the price of replacing τ with τ'. Dropping the primes, we then have established the possibility to choose the **light-cone gauge**:

$$X^+ = \frac{2\pi\alpha'}{l}p^+\tau\,. \tag{5.16}$$

From (3.40) it immediately follows that

$$T_{ab} = 0 \quad \Longleftrightarrow \quad (\partial_\pm X)^2 = 0 \quad \Longleftrightarrow \quad (\dot{X} \pm X')^2 = 0\,, \tag{5.17}$$

where $\dot{X} \equiv \partial_\tau X$ and $X' \equiv \partial_\sigma X$. This implies

$$-2(\dot{X} \pm X')^+(\dot{X} \pm X')^- + (\dot{X} \pm X')^i(\dot{X} \pm X')^i \;=\; 0\,, \tag{5.18}$$

where $(\dot{X} \pm X')^+ = (2\pi\alpha'/l)\,p^+$ by our choice of light-cone gauge. Thus,

$$(\dot{X} \pm X')^- = \frac{(\dot{X} \pm X')^2_\perp}{(4\pi\alpha'/l)\,p^+}\,. \tag{5.19}$$

We see that, up to an additive constant, X^- has ceased to be an independent, dynamical field. To make this manifest, one may Fourier-decompose (5.19) (cf. [11], Ch. 13) to express its oscillator coefficients in terms of the oscillator coefficients of the other fields:

$$\alpha_n^- = \frac{\sqrt{2/\alpha'}}{p^+}\,L_n^\perp \qquad \text{and} \qquad \tilde{\alpha}_n^- = \frac{\sqrt{2/\alpha'}}{p^+}\,\tilde{L}_n^\perp\,. \tag{5.20}$$

Here we have defined

$$L_n^\perp = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_m^i \alpha_{n-m}^i \quad \text{and} \quad \tilde{L}_n^\perp = \frac{1}{2}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_m^i \tilde{\alpha}_{n-m}^i \tag{5.21}$$

with $i \in \{2, \cdots, D-1\}$. In particular, this also applies to the case $n = 0$:

$$p^- = \sqrt{\frac{2}{\alpha'}}\,\alpha_0^- = \sqrt{\frac{2}{\alpha'}}\,\tilde{\alpha}_0^-\,. \tag{5.22}$$

Thus, p^- is also fixed in terms of the other variables.

5.2 Light-cone quantization

It is straightforward to work out the action in light-cone gauge (cf. e.g. [3]):

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\Big((\dot{X}^i)^2 - (X'^i)^2\Big) - \int d\tau\, p^+ \dot{x}^- \,, \tag{5.23}$$

where we have defined

$$x^- \equiv \frac{1}{l}\int_0^{2\pi} d\sigma\, X^-(\tau,\sigma)\,. \tag{5.24}$$

The first term in (5.23) has the usual form. The second term is all that remains of the product X^+X^-. The reason is that X^+ contains just a constant and a linear term in τ. Thus, after taking the partial derivative in τ one gets just a constant. The σ-integration then acts non-trivially only on X^-. We see that x^-, as defined above, is the only independent dynamical part of X^-.

The canonical momenta are

$$p_- \equiv -p^+ \;=\; \frac{\partial L}{\partial \dot{x}^-} \qquad \text{and} \qquad \Pi^i(\sigma) = \frac{1}{2\pi\alpha'}\,\dot{X}^i \;=\; \frac{\partial L}{\partial \dot{X}^i}\,. \tag{5.25}$$

For $i = 2,\ldots,D-1$ this is as before. The mode decomposition of the X^i is also unchanged. Imposing canonical commutation relations for p^+, x^-, and $\Pi^i(\sigma)$, $X^i(\sigma)$ one the has:

$$[x^-,p^+] = -i\;, \qquad [x^i,p^j] = i\delta^{ij}\;, \qquad [\alpha^i_n,\alpha^j_m] = n\,\delta^{ij}\delta_{m+n}\;, \tag{5.26}$$

and analogously for the $\tilde{\alpha_n}^i$ oscillators.

As in old covariant quantization, a key point in the quantization procedure is the normal ordering ambiguity. Here it affects $L_0^\perp$ in (5.20). Formally following the derivation, we find

$$p^+p^- = \frac{1}{2}(p_\perp)^2 + \frac{2}{\alpha'}\frac{1}{2}\sum_{n\neq 0}\alpha^\perp_{-n}\cdot\alpha^\perp_n = \frac{1}{2}(p_\perp)^2 + \frac{2}{\alpha'}\frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}^\perp_{-n}\cdot\tilde{\alpha}^\perp_n\,. \tag{5.27}$$

Here $p_\perp = \sqrt{2/\alpha'}\,\{\alpha_0^i\}$. It is convenient to split the sums in these expressions for p^+p^- into a part which is normal ordered by definition and the normal ordering constant, as we did before:

$$p^+p^- = \frac{1}{2}(p_\perp)^2 + \frac{2}{\alpha'}(N_\perp + a) = \frac{1}{2}\sum(p_\perp)^2 + \frac{2}{\alpha'}(\tilde{N}_\perp + a) \tag{5.28}$$

with

$$N_\perp = \sum_{n>0}\alpha^\perp_{-n}\cdot\alpha^\perp_n \quad \text{and} \quad \tilde{N}_\perp = \sum_{n>0}\tilde{\alpha}^\perp_{-n}\cdot\tilde{\alpha}^\perp_n\,. \tag{5.29}$$

Since, in our conventions,

$$m^2 = 2p^+p^- - (p_\perp)^2, \tag{5.30}$$

we have

$$m^2 = \frac{4}{\alpha'}\sum_n\left(N^\perp + a\right) = \frac{4}{\alpha'}\sum_n\left(\tilde{N}^\perp + a\right). \tag{5.31}$$

Let us now be very explicit about the normal ordering constant:

$$a = \frac{1}{2}\sum_{n\neq 0}\alpha^\perp_{-n}\cdot\alpha^\perp_n - \sum_{n>0}\alpha^\perp_{-n}\cdot\alpha^\perp_n = (D-2)\,\frac{1}{2}\sum_{n<0}(\alpha_n\,\alpha_{-n} - \alpha_{-n}\,\alpha_n). \tag{5.32}$$

Here, in the last expression, we are dealing with a single set of αs, satisfying $[\alpha_{-n}, \alpha_n] = n$ for $n < 0$. Thus,

$$a = (D-2)\,\frac{1}{2}\sum_{n=1}^{\infty} n\,. \tag{5.33}$$

The divergence comes from the UV of our 2d QFT and should be physically understood, regularized, and removed. We will do all that shortly. For now, we just report a shortcut giving the correct result: We regularize the divergent sum using the Riemann zeta function:

$$\sum_{n=1}^{\infty} n \;\to\; \lim_{s\to -1}\sum_{n=1}^{\infty} n^{-s} = \lim_{s\to -1}\zeta(s) = \zeta(-1) = -\frac{1}{12}\,. \tag{5.34}$$

Hence,

$$a = -\frac{D-2}{24}. \tag{5.35}$$

Therefore, we need $D = 26$ for the critical case (i.e. to find a massless photon or graviton), as argued more vaguely before.

As a final comment, we note that we could have made this argument for a at the level of old covariant quantization, but the result would have been wrong. The reason is that our prefactor would have been D, not $(D-2)$. The reason for this necessary reduction is very similar to the photon not having the naive D but only $(D-2)$ physical degrees of freedom. This is apparent in Gupta–Bleuler quantization, which the reader might want to review.

Summary: We have obtained

$$m^2 = \frac{4}{\alpha'}(N_\perp + a) = \frac{4}{\alpha'}(\tilde{N}_\perp + a) \tag{5.36}$$

with

$$a = -\frac{D-2}{24}\,. \tag{5.37}$$

The first equality in (5.36) followed from our expression for p^-, which is not an independent variable and which we expressed in terms of the transverse oscillators and p^+. The second equality is the level matching constraint, arising because we were able to express p^- independently through the left- and right-moving oscillators. Thus, we still have level matching as before:

$$N_\perp = \tilde{N}_\perp. \tag{5.38}$$

Everything is very similar to old covariant quantization, but without longitudinal or '$\pm$' oscillators. Just the physical transverse modes appear. The Fock space construction will therefore have no negative-norm issue.

6 Normal ordering constant and Casimir energy

6.1 Cutoff regularization

Recall that, in light–cone gauge, we had

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\left(\dot{X}_\perp^2 - X_\perp'^2\right) - \int d\tau\, p^+\dot{x}^-, \tag{6.1}$$

with $X_\perp = \{X^2, \cdots, X^{D-1}\}$. The corresponding Hamiltonian follows in the familiar way:

$$H_{lc} = -p^+\dot{x}^- + \int_0^l d\sigma\ \Pi_i \dot{X}^i - \mathcal{L} = \frac{1}{4\pi\alpha'}\int_0^l d\sigma\,\left(\dot{X}_\perp^2 + X_\perp'^2\right). \tag{6.2}$$

It turns out to be a standard QFT Hamiltonian, without any trace of the wrong–sign X^0-field.

Our normal–ordering constant a is, by definition, determined by the vacuum expectation value of this Hamiltonian:

$$E(l) = \langle 0,p|H_{lc}|0,p\rangle = \langle 0,p|\left(\frac{\pi}{l}\sum_{n\neq 0}(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n) + \frac{\pi\alpha'}{l}p_\perp^2\right)|0,p\rangle\,. \tag{6.3}$$

Indeed, apart from the $p_\perp^2$ term, the only source for a possible non-zero value of $E(l)$ is the difference between the oscillator sum above and the corresponding normal-ordered expression. This was our definition of a.

The vaccum energy in (6.3) is divergent. The divergence may be absorbed by a cosmological constant counterterm to the action:

$$S \;\to\; S - \int d^2\xi\,\sqrt{-h}\,\lambda\ , \qquad \text{or, equivalently} \qquad H_{lc} \to\; H_{lc} + l\lambda. \tag{6.4}$$

Now we have

$$E(l) = \frac{\pi\alpha'}{l}p_\perp^2 + \frac{2\pi}{l}(D-2)\sum_{n>0} n + l\lambda\,. \tag{6.5}$$

We regularize by introducing a factor $\exp(-n/l\Lambda)$, where Λ is the cutoff. Then we assume that $\lambda = \lambda(\Lambda)$ and allow it to diverge appropriately to cancel the divergence from the sum over n, rendering E finite.

By our previous definition,

$$a = \frac{D-2}{2}\sum_{n=1}^{\infty} n\,. \tag{6.6}$$

This is of course only meaningful if the regularization and renormalization explained above is implicitly assumed. We can then equivalently write

$$E(l) = \frac{\pi\alpha'}{l}p_\perp^2 + \frac{2\pi}{l}\,2a\,. \tag{6.7}$$

By comparing (6.7) with the Hamiltonian expression (6.5), we see that $4\pi a/l$ is a very physical quantity: It is the Casimir energy of a two–dimensional QFT on a cylinder. The explicit technical result is

$$\frac{4\pi a}{l} = \lim_{\Lambda\to\infty}\left[\frac{2\pi}{l}(D-2)\sum_{n>0} n e^{-n/l\Lambda} + l\lambda(\Lambda)\right]. \tag{6.8}$$

To work this out, it is convenient to introduce $\alpha \equiv 1/l\Lambda$, such that

$$\begin{aligned}
\sum_{n>0}^{\infty} n e^{-\alpha n} &= -\partial_\alpha \sum_{n>0}^{\infty} e^{-\alpha n} = -\partial_\alpha \frac{1}{1-e^{-\alpha}} = \frac{e^{-\alpha}}{(1-e^{-\alpha})^2} = \frac{1-\alpha+\frac{\alpha^2}{2}}{\left(\alpha-\frac{\alpha^2}{2}+\frac{\alpha^3}{6}\right)^2} + \mathcal{O}(\alpha) \\
&= \frac{1}{\alpha^2}\frac{1-\alpha+\frac{\alpha^2}{2}}{\left(1-\frac{\alpha}{2}+\frac{\alpha^2}{6}\right)^2} + \mathcal{O}(\alpha) = \frac{1}{\alpha^2}\frac{1-\alpha+\frac{\alpha^2}{2}}{1-\alpha+\frac{7\alpha^2}{12}} + \mathcal{O}(\alpha) = \frac{1}{\alpha^2}\frac{1+\frac{\alpha^2}{2}}{1+\frac{7\alpha^2}{12}} + \mathcal{O}(\alpha) \\
&= \frac{1}{\alpha^2}\left(1-\frac{1}{12}\alpha^2\right) + \mathcal{O}(\alpha) = (\Lambda l)^2 - \frac{1}{12} + \mathcal{O}(\frac{1}{l\Lambda}),
\end{aligned} \tag{6.9}$$

where the divergent $(\Lambda l)^2$ term can be cancelled by a counterterm $\lambda(\Lambda) \sim \Lambda^2$. Crucially, this $\lambda(\Lambda)$ counterterm has no chance to influence the decisive constant $(-1/12)$ because of the wrong l-dependence. This term is a UV-insensitive observable in 2d QFT, determining the Casimir energy. Therefore,

$$a = (D-2)\left(-\frac{1}{12}\right) = -\frac{D-2}{24} \equiv -1 \quad \text{for} \quad D = 26. \tag{6.10}$$

6.2 Regularization independence

We have argued on physical grounds, in terms of a QFT on cylinder, that the normal ordering constant a is physical and hence regularization-independent. More precisely, the Casimir energy scales as $1/l$ for dimensional reasons. The cosmological constant counterterm gives a contribution $\sim l\Lambda^2$ and thus cannot affect the Casimir energy. Based on this logic, any smooth cutoff should work, e.g.

$$e^{-n/l\Lambda} \quad \longrightarrow \quad f\left(\frac{n}{l\Lambda}\right), \qquad \text{with} \quad f\to 0 \quad \text{as} \quad \Lambda\to\infty\,. \tag{6.11}$$

Demonstrating this explicitly is in general nontrivial – cf. the discussion of the $4d$ case in [12], Sec. 3.2.4.

Let us now sketch how dimensional regularization would work. The standard QFT expression for the vacuum energy is

$$E \;\sim\; V_{d-1}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\cdot\left(\frac{1}{2}\,\omega_p\right); \qquad \omega_p = \sqrt{\overline{p}^2+m^2}\,, \tag{6.12}$$

with $\overline{p} \in \mathbb{R}^{d-1}$. Let $m = 0$ and compactify one spatial dimension on a circle of circumference $l = 2\pi R$. Then one component of $\overline{p}$ becomes discrete and we find

$$E \;=\; \frac{1}{2}\,2\pi R\, V_{d-2}\,\frac{1}{2\pi R}\sum_n \int \frac{d^{\,d-2}p}{(2\pi)^{d-2}}\,\sqrt{\frac{n^2}{R^2}+\overline{p}^2}\;. \tag{6.13}$$

Now $\overline{p} \in \mathbb{R}^{d-1}$ and we have used the replacement

$$\int dp \;\to\; \frac{1}{2\pi R} \sum_n \,. \tag{6.14}$$

We note that

$$\int \frac{d^{d-2}p}{(2\pi)^{d-2}} \sqrt{\left(\frac{n}{R}\right)^2 + \overline{p}^2} \quad \sim \quad \left(\frac{n}{R}\right)^{d-1} \tag{6.15}$$

on dimensional grounds. From this it follows immediately that in our 2-dimensional case dimensional regularization corresponds to replacing

$$\sum_{n\neq 0} n \quad \to \quad \sum_{n\neq 0} |n|^{\,d-1} \tag{6.16}$$

and taking the limit $d \to 2$. This is precisely what we did when applying ζ-function regularization.

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