10. Conformal Field Theory - Basics

(See e.g. Blumenhagen / Plauschinn ; DiFrancesco / Mathieu / Senechal)

• Motivation: We arrived at a QFT defined by

$$Z = \int DX\, Db\, Dc\, \exp(-S_X - S_{FP}).$$

Due to the still unfixed residual gauge freedom, this theory has a large symmetry which, if we for the moment ignore our desire to gauge it, make it a conformal field theory or CFT. It is worthwhile to step back and study CFTs in their own right.

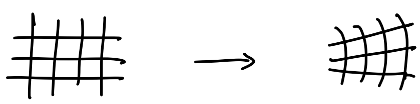
10.1 Conformal trfs. in general dimensions

• Conf. trf.: A diffeomorphism, $x \to x'(x)$, under which

$$g'_{\mu\nu}(x') \equiv g_{\alpha\beta}(x) \cdot \left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right) \overset{!}{=} g_{\mu\nu}(x) \underbrace{e^{2\omega(x)}}_{}$$

change only by overall factor.

• Equivalently: Angle-preserving diffeomorphisms



• Infinitesimally: (cf. our earlier 2d calculations)

If $x'^\mu = x^\mu + \epsilon^\mu(x)$ and $g_{\mu\nu} = \eta_{\mu\nu}$, we need

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\omega \eta_{\mu\nu} \underset{d>1}{\Longrightarrow} (2-d)\partial_\mu\partial_\nu \omega = 0$$

$$\& \ \partial^2 \omega = 0$$

(cf. problems)

- It follows (for $d > 2$) that ω is at most a linear fct. Hence ϵ is at most quadratic and the group of conf. trfs. is <u>finite-dimensional.</u>

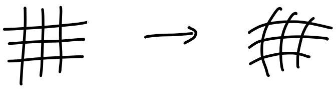
- The generators of conf. trfs. in $\mathbb{R}^d$ are:

  (1) Poincare trfs. – $\binom{d}{2}$ parameters

  (2) dilatation – $x'^\mu = \alpha x^\mu$ – 1 parameter

  (3) special conformal trfs.: $x'^\mu = \dfrac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2}$, d parameters

  [The last can be realized as "Inversion $\circ$ Translation $\circ$ Inversion" with Inversion $= \{x^\mu \to x^\mu / x^2\}$. Locally, one may think of the change in geometry caused by (3) as



  ]

- <u>Facts:</u> Group of conf. trfs. in $\mathbb{R}^d$ $= SO(1, d+1)$

  – " – in $\mathbb{R}^{1,d-1}$ $= SO(2, d)$

10.2 Conformal trfs. in $d = 2$

- As you have seen in the problems, the derivation of $\partial_\mu \partial_\nu \omega = 0$ fails in $d = 2$. But $\partial^2 \omega = 0$ still holds and (as we know from our discussion of residual gauge freedom) is solved by

$$\epsilon^+ = \epsilon^+(\sigma^+) \; ; \; \epsilon^- = \epsilon^-(\sigma^-) \qquad \text{(in light-cone coordinates)}$$

$\Rightarrow$ The group of conf. trfs. is <u>infinite-dimensional</u>.

- In modern string theory, calculations are always done in the euclidean WS theory (trusting in the analytic continuation $\tau \to -i\tau$, as in QFT). Then $e^{iS} \to e^{-S}$, with $S \sim \int (\partial\phi)^2$ with pos.-definite metric.

- For a euclidean WS, the transition to $\sigma^\pm$ is not helpful. Instead, the metric $ds^2 = \delta_{ab}\, d\xi^a d\xi^b$ simplifies in the coordinates

$$z = \xi^1 + i\xi^2 \,, \quad \bar{z} = \xi^1 - i\xi^2 \,,$$

where $ds^2 = dz\, d\bar{z}$ or $h_{z\bar{z}} = h_{\bar{z}z} = \frac{1}{2}$, $h_{zz} = h_{\bar{z}\bar{z}} = 0$.

- Now $\partial_a \epsilon_b + \partial_b \epsilon_a = 2\omega\delta_{ab}$ implies $\partial_z \epsilon_z = 0$ & $\partial_{\bar{z}} \epsilon_{\bar{z}} = 0$.

$\Rightarrow \epsilon^z = \epsilon^z(z)$; $\epsilon^{\bar{z}} = \epsilon^{\bar{z}}(\bar{z})$ $\quad [\omega = \partial_z \epsilon_{\bar{z}} + \partial_{\bar{z}} \epsilon_z]$

- $\epsilon^{\bar{z}}(\bar{z}) = \overline{\epsilon^z(z)}$, to ensure that $\epsilon^1 = (\epsilon^z + \epsilon^{\bar{z}})/2$ & $\epsilon^2 = (\epsilon^z - \epsilon^{\bar{z}})/2i$ remain real. To simplify notation, one writes $\epsilon^z(z) \equiv \epsilon(z)$; $\epsilon^{\bar{z}}(\bar{z}) = \bar{\epsilon}(\bar{z})$.

<u>Summary:</u> • We think of our 2d space as $\mathbb{R}^2 = \mathbb{C}$.
(More generally: 1d complex manifold or "Riemann surface".)

• Conf. trfs. are <u>holomorphic trfs.</u>:

- infinitesimal: $\quad z \to z + \epsilon(z)$; $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$
- finite: $\quad z \to z' = w(z)$; $\bar{z}' \to \bar{z}' = \bar{w}(\bar{z})$.

10.3 Conformal field theories

- Let us now consider fields on our space or spacetime.

- In our 2d euclidean case (but you are invited to think of this more generally), scalars transform as:

$$\delta\phi = -\epsilon^a \partial_a \phi = (-\epsilon(z)\partial_z + \bar{\epsilon}(\bar{z})\partial_{\bar{z}})\phi$$

- If we think of this as a reparametrization (i.e. change the metric by the standard formula), invariance is trivial for any reasonable QFT. The key point is that, for a CFT, we demand invariance of theory under trfs. $\delta\phi$ as above at fixed metric. In other words, we transform the field configuration on a fixed space with fixed metric by a conformal trf. acting just on the fields. This invariance is highly non-trivial and requires, in particular, that our QFT "does not see" the scale trf. by $e^{2\omega}$ which we may think of as "undoing" the metric change induced by the diff.

- Explicitly, we may think of ϵ in terms of its Laurent series:

$$\epsilon = \sum_{n=-\infty}^{\infty} c_n z^n \quad \Rightarrow \quad \delta\phi = \sum_{n=-\infty}^{\infty} (c_{n+1} \ell_n + \bar{c}_{n+1} \bar{\ell}_n)\phi$$

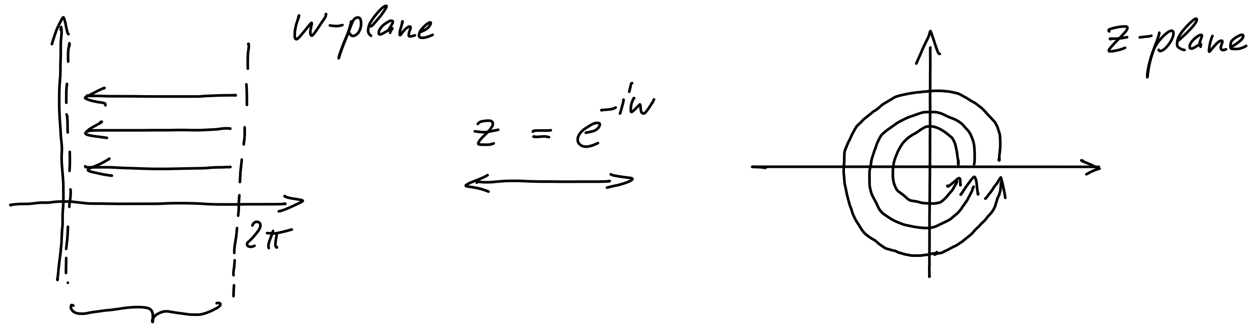
$$\text{with} \quad \underbrace{\ell_n \equiv -z^{n+1}\partial_z \; ; \; \bar{\ell}_n \equiv -\bar{z}^{n+1}\partial_{\bar{z}}}_{\text{These are two sets of Witt algebra generators. (Check this!)}}$$

10.4 The String WS CFT

- We are interested in the QFT with fields X, b, c and $S = S_X + S_{FP}$. This is diff-invariant (obviously) and Weyl invariant (at least classically). Hence, by our explanations of the last section it is a CFT:

  | Again, apply a standard diff with the restriction that h_{ab} is only changed by a scale factor. Undo this rescaling by "Weyl" $\Rightarrow$ This is our conf. trf. acting on the QFT. |

- To develop some intuition, consider a particularly important trf. for 2d CFTs and particularly in the string context:



closed WS with width 2π Note: In the z-plane, time grows radially

- The EM-tensor is important in CFTs in general. By replacing our earlier σ^+/σ^- analysis with an analogous $z/\bar{z}$ analysis, we have:

T_{ab} traceless $\Rightarrow$ $T_{z\bar{z}} = 0$

T_{ab} conserved $\Rightarrow$ $\partial_{\bar{z}} T_{zz} = 0$ & $\partial_z T_{\bar{z}\bar{z}} = 0$

- For notational simplicity, define $T \equiv T_{zz}$; $\bar{T} \equiv T_{\bar{z}\bar{z}}$.
  (Note: $\bar{T}$ is <u>not</u> the complex-conjugate of T.)

- Our key CFTs: $S = \frac{1}{2\pi\alpha'}\int d^2z\,(\partial X)(\bar\partial X)$ & $S = \frac{1}{2\pi}\int d^2z\; b\bar\partial c$

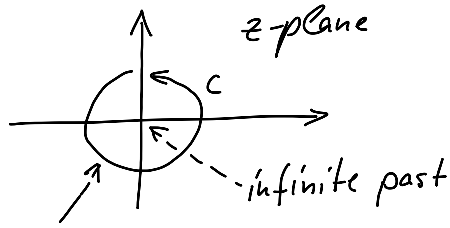
- Solutions: $X(z,\bar z) = \underset{\uparrow}{X(z)} + \underset{\uparrow}{\bar X(\bar z)}$ $\qquad\left[\bar\partial \equiv \frac{\partial}{\partial \bar z}\right]$

Again, these are two independent fcts.!

$$\Rightarrow \quad T = -\frac{1}{\alpha'}(\partial X)(\partial X)\;,\quad \bar T = -\frac{1}{\alpha'}(\bar\partial\bar X)(\bar\partial \bar X)$$

- The L_m are defined precisely as before. But we now give the eqs. in the z-frame (not the more familiar w-frame):

$$L_m = \frac{1}{2\pi i}\oint_C \frac{dz}{z}\, z^{m+2}\, T(z)$$



corresponds to going once across the cylinder

from $w \sim \ln z$
$dw \sim dz/z$

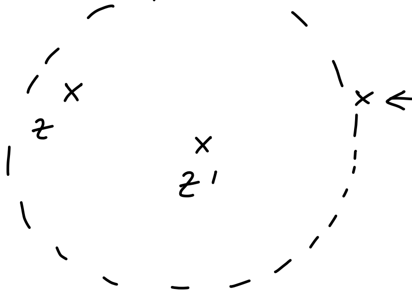
from $T_{zz} = T_{ww}\left(\frac{\partial w}{\partial z}\right)^2 + \underbrace{\cdots}_{\text{see later}}$

10.5 Operator product expansion (OPE)

- Let $O_i(z,\bar z)$ be the set of all local operators, such as X, ∂X, $X\cdot X$ etc.

- Key OPE claim:

$$O_i(z,\bar z)\, O_j(z',\bar z') = \sum_k C_{ij}{}^k(z-z', \bar z - \bar z')\, O_k(z',\bar z')$$



← nearest further operator defines radius of convergence.

[The OPE is to be thought of "under the path integral" or in a vacuum expectation value, $\langle \cdots \rangle$, with other operators inserted.]

- We will view the OPE simply as a "Laurent series expansion with operator coefficients". Note, however, that 2d CFT is in principle a mathematically well-defined subject and the OPE can be proven.

10.6 Ward identities

(We now follow D. Tong's lecture notes)

- Let $\phi \to \phi' = \phi + \epsilon\delta\phi$ be a global symmetry (i.e. S & $D\phi$ are invariant)
- Now consider the same trf. with $\epsilon = \epsilon(x)$.

δS will be non-zero thanks to the ϵ not being constant.

$$\Rightarrow \quad \delta S \sim \partial_a \epsilon \quad \text{or, more precisely,} \quad \delta S = \underset{\substack{\uparrow \\ \text{Stringy convention}}}{\frac{1}{2\pi}} \int J^a \partial_a \epsilon$$

- One easily sees that J is conserved in the quantum theory:

$$0 = \int \underbrace{D\phi'}_{= D\phi} e^{-S[\phi']} - \int D\phi\, e^{-S[\phi]} \qquad \text{with } \phi' \equiv \phi + \epsilon\delta\phi$$

$$\Rightarrow \quad 0 = \int D\phi \left(e^{-S[\phi] - \delta S[\phi]} - e^{-S[\phi]} \right) = \int D\phi\, e^{-S[\phi]} \delta S$$

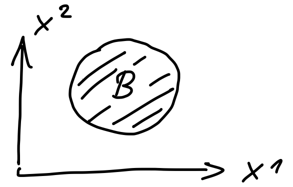
$$= \frac{1}{2\pi} \int D\phi\, e^{-S[\phi]} (-\partial_a J^a) \cdot \epsilon(x) \qquad \text{for any fct. } \epsilon(x)$$

$$\Rightarrow \quad \langle \partial_a J^a \rangle = 0$$

- Now we repeat the analysis with a local operator $O(x_1)$ under path integral. Let $O' = O + \epsilon\delta O$ under our symm. trf.

$$0 = \int D\phi \left(e^{-S-\delta S} (O + \epsilon\delta O) - e^{-S} O \right) = \cdots = \int D\phi\, e^{-S} \left(\epsilon\delta O + \frac{1}{2\pi} \int_x \partial_a J^a O \epsilon \right)$$

Let ϵ be non-zero only in a ball B:



$$\Rightarrow \langle \delta O(x_1) \cdots \rangle = -\frac{1}{2\pi} \int_B d^d x \, \partial_a \langle j^a(x) O(x_1) \cdots \rangle$$

any other operators can be inserted here if they are outside B.

- The r.h. side is zero if x_1 is outside B by current conservation.
- If x_1 is inside B, we get

$$\int_B d^d x \, \partial_a \langle j^a \cdots \rangle = \int_{\partial B} d^{d-1} y \, \hat{n}^a \langle j_a \cdots \rangle$$

normal vector of surface

- Specifically in $d=2$, this becomes $\int_{\partial B} d\vec{f} \cdot \vec{j}$

$$= \int \begin{pmatrix} dx^2 \\ -dx^1 \end{pmatrix} \cdot \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} = \int (dx^2 j_1 - dx^1 j_2) \; ; \quad dz = dx^1 + i dx^2$$

$$= -i \oint (dz \, j_z - d\bar{z} \, j_{\bar{z}}) \qquad j_z = \tfrac{1}{2}(j_1 - i j_2)$$

$$\Rightarrow \boxed{\delta O(z_1) = \frac{i}{2\pi} \oint \left(dz \, j_z(z,\bar{z}) - d\bar{z} \, j_{\bar{z}}(z,\bar{z}) \right) O(z_1)}$$

To be thought of as being under the path integral (i.e. in $\langle \cdots \rangle$), with no other operator being inside the contour around z_1.

10.7 Conformal Ward identity

- Let us now consider conformal trfs., i.e. $\delta z = \epsilon(z)$, as a global symm. of a 2d CFT. (Note that this a global symm., in the sense of not being gauged, in spite of ϵ being a function. Each holom. fct. $\epsilon(z)$ corresponds to one independent global symm.)

- Note that we do not reparameterize our space but move the fields, i.e. $\delta\phi = \partial_z \phi\, \delta z + \partial_{\bar z}\phi\, \delta \bar z$. If we were to change the metric correspondingly, the action would clearly be invariant. Thus, any possible change of S can be obtained by just changing the metric:

$$\delta S = \int d^2\xi \frac{\delta S}{\delta h_{ab}} \delta h_{ab} \quad , \text{ with } \delta h_{ab} = -\partial_a \epsilon_b - \partial_b \epsilon_a$$

$$\Rightarrow \delta S = \frac{1}{2\pi}\int d^2\xi\, T_{ab}\, \partial^a \epsilon^b \underset{\uparrow}{=} \frac{1}{\pi}\int d^2 z \left(T_{zz}\, \partial_{\bar z} \epsilon + T_{\bar z \bar z}\, \partial_z \bar\epsilon\right)$$

$T_{z\bar z} = T_{\bar z z} = 0$ by tracelessness

- This is zero if ϵ is holomorphic, confirming that holomorphic tfs. are global symmetries.

- Now we replace $\epsilon(z) \longrightarrow \epsilon(z)\cdot f(\bar z)$

corresponds to constant ϵ of last section — corresponds to non-constant ϵ of last section.

$$\Rightarrow \delta S = \frac{1}{2\pi}\int d^2 z\, 2 \underbrace{\left[T_{zz}\, \epsilon(z)\, \partial_{\bar z} f(\bar z) + T_{\bar z\bar z}\, \bar\epsilon(\bar z)\, \partial_z \bar f(z)\right]}_{\hat{=}\ J^a \partial_a \epsilon \text{ of last section.}}$$

$$\Rightarrow J^{\bar z} = 2 T_{zz}(z)\epsilon(z) \quad , \text{ i.e. } \boxed{J_z = T(z)\,\epsilon(z)}$$

- Since $J_z / J_{\bar z}$ are holomorphic / anti-holomorphic, our previous formula for δO in terms of a contour integral gives simply the residues:

$$\boxed{\delta O(w,\bar w) = -\operatorname{Res}_{z\to w}\left[\epsilon(z) T(z) O(w,\bar w)\right] - \operatorname{Res}_{\bar z \to \bar w}\left[\bar\epsilon(\bar z)\bar T(\bar z) O(w,\bar w)\right]}$$

- This is a key result. It tells us that the OPE of with T calculates the conf. trf. of any operator O in the full quantum theory.

- A more intuitive explanation ($\to$ Polchinski) goes as follows:

  - $\delta O \sim [Q, O] \sim QO - OQ$.
  - Let O be the generator of our symm. (e.g. $\in T$) and think of the operators in the Heisenberg picture in the radial frame.
  - Then: $\delta O \sim$