12 Scattering amplitudes

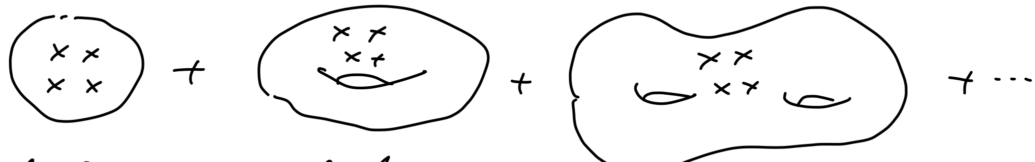
12.1 String coupling and sum over Riemann surfaces

Recall the basic definition of a string amplitude given earlier:

$$\mathcal{A} = \sum_{g=0}^{\infty} \int \frac{Dh\, DX}{\mathrm{Vol}_{\mathrm{Diff} \times \mathrm{Weyl}}} e^{-S[X,h]} \underbrace{\int d^2\xi_1 \cdots \int d^2\xi_n}\, V_1(\xi_1) \cdots V_n(\xi_n)$$

"genus" g

(in our earlier discussion we did not make this integration manifest)

Illustration: 

$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

- The "1/Vol" has been introduced to cancell the "Vol" from the integral over the gauge group.

- In our earlier discussion, we have ignored a key term in S:

$$S[X,h] = \underbrace{\frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \sqrt{h} (\partial X)^2}_{S_X,\text{ as before}} + \underbrace{\frac{1}{4\pi} \int_\Sigma d^2\xi \sqrt{h}\, \phi R + \frac{1}{2\pi} \int_{\partial\Sigma} ds\, \phi K}_{S_R}$$

- ϕ is part of the target-space background, just like $\eta_{\mu\nu}$ (which in principle could be any metric $g_{\mu\nu}$ and the fluctuations of which are the graviton we found at level 1.

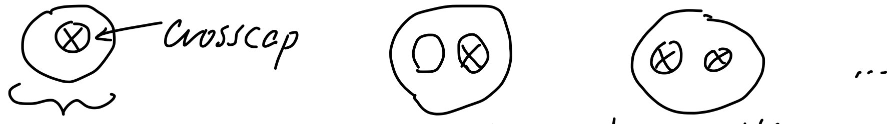
at constant ϕ, which we will always assume, this is a purely topological term determined by the Euler number of the surface: $S_R = \phi \cdot \chi(\Sigma)$.

$$\chi(\Sigma) = 2 - 2g - b - c$$

genus (g); # of boundaries & crosscaps (b, c)

- The fluctuations of ϕ are our extra scalar or "<u>dilaton</u>"

- Examples:



real projective plane   Möbiusstrip   Klein bottle

- Define $g_s \equiv e^{\phi}$
- Focussing on closed-string amplitudes, we see that every extra "hole" ("loop", in QFT language), produces an extra factor g_s^2 in $\mathcal{A}$.

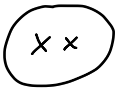
$\Rightarrow$ At $\phi \to -\infty$, our theory becomes weakly coupled

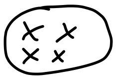
- The normalization of the V's is to some extent conventional since it is related to the normalization of the corresponding target-space fields. One usually includes a factor g_s in V. Thus, the tachyon vertex operator is

$$V(k, z, \bar{z}) = g_s : e^{ik^\mu X_\mu(z,\bar{z})} :$$

(with $(z,\bar{z}) \hat{=} \xi$)

- As a result, one has for the 2-pt-fct.,

 $\sim g_s^2 \cdot e^{-\phi(2-2g)} \sim g_s^2 g_s^{-2} \sim 1$,

and then additional g_s-factors for loops (), 4-pt.-fcts. (), etc.

12.2 Gauge fixing

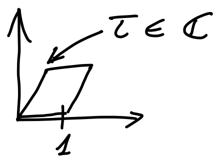
- We thus have: $\mathcal{A} \sim g_s^{2g-2} \int \frac{DX Dh}{Vol_{Diff \times Weyl}} \cdot e^{-S_X} \prod_i \int_{z_i} V_i$

- Our earlier path integral discussion showed, at a rough level

$$\int Dh = Vol_{Diff \times Weyl} \cdot \Delta_{FP}[\hat{h}]$$

- Now we need to include two finer effects, which were already mentioned before:

a) On a compact Riemann surface, the metric generically includes d.o.f. which can not by absorbed in Diff × Weyl. An example are the shape moduli of a torus:



Let us denote these moduli by

$$t^k \; ; \; k = 1, \dots, \mu \qquad \text{s.t.} \quad \hat{h} = \hat{h}(t)$$

b) There are gauge trfs. not changing the metric (conformal Killing vectors). On a compact space there will be finitely many, let's call their number κ. Example: $SL(2,\mathbb{C})$ trfs. of S^2. If enough Vertex operators are present (as we will assume) then the positions of κ of those can be fixed using the residual gauge freedom.

- We now proceed by treating the integral over the metric together with κ of the Vertex integrations:

$$\frac{1}{\mathrm{Vol}_{D\times W}} \int Dh \prod_{i=1}^{\kappa} d^2\xi_i = \frac{1}{\mathrm{Vol}_{D\times W}} \int Dh \, \underbrace{D\zeta}_{\text{Diff}\times\text{Weyl}} \, d^\mu t \prod_{i=1}^{\kappa} \int d^2\xi_i \, \delta(\xi_i - \underbrace{\hat{\xi}_i^\zeta}_{\text{fixed Vertex positions}}) \, \delta[h - \hat{h}^\zeta(t)] \, \Delta_{FP}$$

Here the ζ & t-integrations remove all δ-fcts. and Δ_{FP} compensates the inverse determinant appearing in this process. We now carry out h and ξ_i-integrations and eventually the ζ-integration:

$$\dots = \frac{1}{\mathrm{Vol}_{D\times W}} \int D\zeta \, d^\mu t \, \Delta_{FP} = \int d^\mu t \, \Delta_{FP} \, .$$

We explicitly have:

$$\Delta_{FP}^{-1} = \int d^\mu t \int D\zeta \, \delta[h - \hat{h}^\zeta(t)] \prod_i \delta(\xi_i - \hat{\xi}_i^\zeta) .$$

- Note that ζ and t define <u>together</u> our FP determinant.

  The two relevant descriptions are $\hat{\xi}_i, h \underset{\Delta_{FP}}{\longleftrightarrow} \zeta, t$.

- <u>Intuitively:</u> $\hat{h}^{\zeta}(t)$ parameterizes all metrics. But ζ, t have slightly more freedom – the residual gauge trfs. The match between ζ, t and h hence only becomes perfect when supplementing h with the κ points $\hat{\xi}_i$.

- We now make manifest how $\hat{h}^{\zeta}(t)$ changes due to ζ (infinitesimally given by ϵ & ω) and t (infinitesimally given by δt):

$$\delta \hat{h}_{ab} = -(P\epsilon)_{ab} + (2\omega - D_c \epsilon^c)\hat{h}_{ab} + \sum_{k=1}^{\mu} \delta t^k \, \partial_{t^k} \hat{h}_{ab}(t)$$

- Analogously, $\hat{\xi}_i^{\zeta}$ changes due to ζ as $\delta \hat{\xi}_i^{\zeta} = \epsilon(\hat{\xi}_i)$.
- Thus, we have

$$\Delta_{FP}^{-1} = \int d^{\mu}\delta t \; D\epsilon \int d^{\kappa}x \; D\beta \; \exp 2\pi i \left[\int \beta \cdot \left(-P\epsilon + \delta t^k \partial_k \hat{h}\right) + x_i \cdot \epsilon(\hat{\xi}_i) \right]$$

Integrations realizing δ-fcts.

infinitesimal versions of t, ζ integration above. The ω-integration has been carried out leaving β traceless by definition.

- Now, to get Δ_{FP}, we simply make all our integration variables Grassmann. $\epsilon \to c$; $\beta \to b$; $x \to \eta$; $\delta t \to \varsigma$

  all Grassmann variables / fields.

- The variables η, ς appear only linearly and w/o derivatives in the exponent. So we can perform the integration giving

$$\prod_{\text{fixed } i} c^1(\hat{\xi}_i) \cdot c^2(\hat{\xi}_i) \qquad \text{and} \qquad \prod_{k=1}^{\mu} \int b \cdot \partial_k \hat{h} \, .$$

- The rest produces the usual FP-action together with the b, c integrations. Thus, collecting all the factors from our previous formula for the amplitude, we have:
(changing notation according to $\xi \to z, \bar{z}$)

$$\mathcal{A}_n = \sum_g g_s^{-2+2g} \int d^\mu t\, DX\, Db\, Dc\, e^{-S_X - S_{FP}} \prod_{\text{unfixed } z_i} \int d^2 z_i \prod_{\text{fixed } z_i} c^1(\hat{z}_i) c^2(\hat{z}_i)$$
$$\prod_{k=1}^{\mu} \int b \cdot \partial_k h \prod_{i=1}^{n} \sqrt{h(z_i)}\, V_i(z_i).$$

12.3 The Virasoro-Shapiro amplitude

(4-tachyon scattering at tree-level, i.e. on S^2)

- Facts: - S^2 has no moduli
  - Conf. trfs. on S^2 allow one to move 3 arbitrary points to 3 desired fixed positions (e.g. $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$).

[cf. BLT; Ginsparg's lecture notes]

$$\Rightarrow \mathcal{A}_4 = g_s^2 \int d^2 z_4 \left\langle :e^{ik_4 X(z_4, \bar{z}_4)}: \prod_{i=1}^{3} :c(\bar{z}_i)\bar{c}(\bar{z}_i) e^{ik_i \cdot X(z_i, \bar{z}_i)}: \right\rangle$$

- Obviously, the X & $c\bar{c}$ - parts split.
- Ignoring normal ordering, the X-part reads

$$\int DX \exp\left[\frac{1}{2\pi\alpha'} \int d^2 z\, X \partial\bar{\partial} X + i \int d^2 z\, J \cdot X\right]$$

with $J(z, \bar{z}) \equiv \sum_{i=1}^{4} k_i\, \delta^2(z - z_i, \bar{z} - \bar{z}_i)$

- Split the X-integral as

$$\int \prod_{\mu=1}^{26} DX^\mu = \int \prod_{\mu=1}^{26} D\tilde{X}^\mu \int d^{26} x \,,$$ where $\tilde{X}$ contains only the oscillators, not the zero mode.

- The zero-mode part gives $\int d^{26}x \, e^{i\sum_{i=1}^{4} k_i \cdot x} \sim \delta^{26}\left(\sum_{i=1}^{4} k_i\right)$

- The non-zero-mode part is easily evaluated since the operator in the quadratic piece of the exponent is invertible:

$$\Rightarrow \quad \sim \exp\left(\frac{\pi\alpha'}{2}\int d^2z\, d^2z' \, J(z,\bar z)\frac{1}{\partial\bar\partial}J(z',\bar z')\right).$$

- We may use $\frac{1}{\partial\bar\partial} = \frac{1}{\pi}\ln|z|$ (although this is strictly speaking too naive because on a compact space one needs to introduce a background charge to be able to solve the KG-eq. with δ-source. Neverthelen, the naive approach gives the correct result. See Polchinski, Sect. 6.2 for details).

- $\Rightarrow \quad \sim \exp\left(\frac{\alpha'}{2}\sum_{ij} k_i \cdot k_j \ln|z_i - z_j|\right)$

- Now we recall that normal ordering is present, s.t. no contributions with $i=j$ arise. $\Rightarrow \quad \sim \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}$

- Next, the ghost part: $\langle c(z_1)c(z_2)c(z_3)\rangle$ & $\langle \bar c(\bar z_1)\bar c(\bar z_2)\bar c(\bar z_3)\rangle$

- First, note the (maybe counterintuitive) fact that these 3-pt-fcts are $\neq 0$. Technical reason (cf. Polchinski 5.3): 3 conf. Killing vectors $\to$ 3 corresponding zero modes of path integral $\to$ need 3 insertions for non-zero result due to Grassmann nature of int.

- Completely generally ($\to$ e.g. Ginsparg, BLT, ...): For primary fields, conf.-symm. fixes z-dependence of 2 & 3-pt. fcts.:

$$G^{(3)} \sim C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{13}^{h_1+h_3-h_2}} \cdot \frac{1}{\left\{ \begin{array}{c} z \to \bar{z} \\ h \to \bar{h} \end{array} \right\}}$$

to be calculated (→ C_{123})

$$z_{ij} \equiv z_i - z_j$$

- For c-ghost: $h = -1$, $\bar{h} = 0$ (clear since c is fermionic version of vector ϵ, as opposed to co-vector ∂X.)

$$\Rightarrow \langle c(z_1) c(z_2) c(z_3) \rangle \sim z_{12} z_{23} z_{13} \quad (\text{\& analogously for } \bar{c})$$

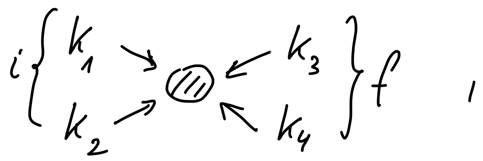
$\Rightarrow$ z-dependence coming from X & $c\bar{c}$ together:

$$\sim |z_{12}|^{\alpha' k_1 k_2 + 2} |z_{13}|^{\alpha' k_1 k_3 + 2} |z_{23}|^{\alpha' k_2 k_3 + 2} |z_{14}|^{\alpha' k_1 k_4} |z_{24}|^{\alpha' k_2 k_4} |z_{34}|^{\alpha' k_3 k_4}$$

- Using conf. symm. on S^2 (i.e. $SL(2,\mathbb{C})$) to move 3 vertex operators to

$$(z_1, z_2, z_3) = (0, 1, \infty)$$

- Use Mandelstam varicbles:



$$s = -(k_1 + k_2)^2 = -2k_1 k_2 + 2m^2$$
$$t = -(k_1 + k_3)^2 = -2k_1 \cdot k_3 + 2m^2$$
$$u = -(k_1 + k_4)^2 = -2k_1 k_4 + 2m^2 .$$

- In complete generality,

$$s + t + u = \sum_{i=1}^{4} m_i^2 . \quad \xrightarrow{\text{our case}} \quad s + t + u = 4m^2$$

- Now return to the z-dependent factor in $\mathcal{A}_4$ given above. Focus only on the terms involving z_3. We have:

$$\alpha'(k_1 k_3 + k_2 k_3 + k_3 k_1) = \frac{\alpha'}{2}(-s - t - u + 6m^2) = \alpha' m^2 = -4$$

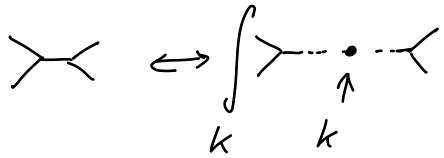
$\Rightarrow$ The product of the three terms with z_3 goes to 1 as $z_3 \to \infty$.

- Rename $z_4 \to z$; Reinstate z-integration etc. ... $\Rightarrow$

$$\mathcal{A} = i g_s^2 C_{S^2} (2\pi)^{26} \delta^{26}\Big(\sum_{i=1}^{4} k_i\Big) \int d^2 z \, |z|^{-\frac{\alpha'}{2}u - 4} \, |1-z|^{-\frac{\alpha'}{2}t - 4}$$

$\uparrow$

[normalization factor ; can be determined by explicit evaluation of the determinants arising in path integrals on sphere or from unitarity, e.g.



; cf. Polchinski]

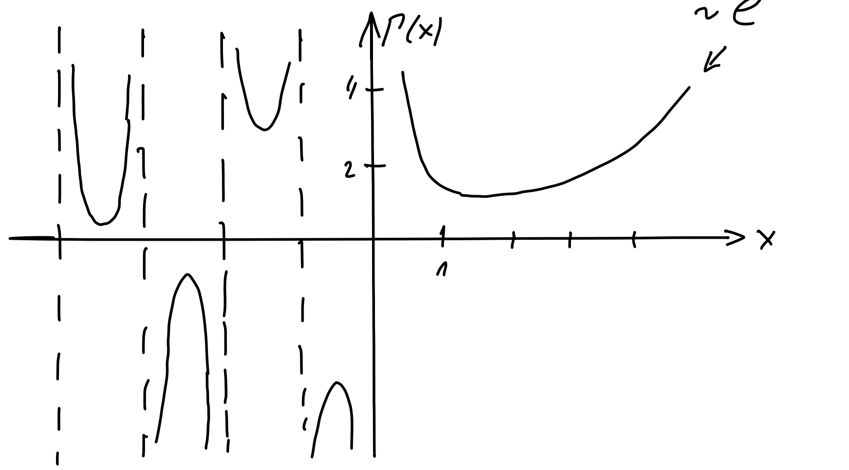
- Define: $C(a,b,c) \equiv \int d^2 z \, |z|^{2a-2} \, |1-z|^{2b-2}$; $c \equiv 1-a-b$

Fact: $$C(a,b,c) = \frac{2\pi \, \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}$$ ($\to$ Tong, Polch., GSW problems)

$$\Rightarrow \mathcal{A}_4 = i g_s^2 C_{S^2} (2\pi)^{26} \delta^{26}(\ldots) \, \frac{\Gamma(-1-\frac{\alpha' s}{4})\Gamma(-1-\frac{\alpha' t}{4})\Gamma(-1-\frac{\alpha' u}{4})}{\Gamma(2+\frac{\alpha' s}{4})\Gamma(2+\frac{\alpha' t}{4})\Gamma(2+\frac{\alpha' u}{4})}$$

(famous result!)

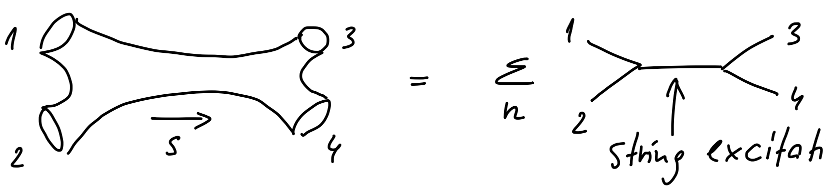
- Recall Γ-fct.:



outside poles ($x \notin \mathbb{Z}$), can use Euler's reflection formula: $\Gamma(1-x)\Gamma(x) = \pi / \sin \pi x$.

- We see: $\mathcal{A}_4$ has many poles.

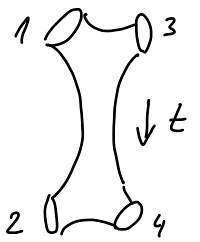
e.g. from <u>s-channel exchange</u>:



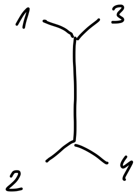
string excitations at level n with $M_n^2 = \frac{4}{\alpha'}(n-1)$.

These effects correspond to poles in $\Gamma(-1-\frac{\alpha' s}{4})$ at $s = M_n^2$.

- $\mathcal{A}_4$ also has poles from <u>t-channel exchange</u>: (or u-channel exchange)



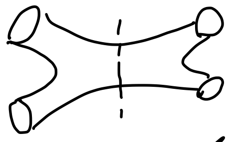
$= \sum_n$



$\leftarrow$ poles from $\Gamma(-1-\frac{\alpha' t}{4})$.

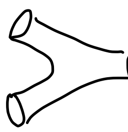
- In contrast to QFT, all these poles come from the same WS. No need to sum different diagrams, like in QFT.

- <u>Comment</u>: C_{S^2} can be fixed since

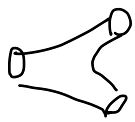


near pole

can be related to



$\cdot$



. The former is $\sim C_{S^2}$, the latter to $(C_{S^2})^2$. $\Rightarrow$ C_{S^2} can be determined.

- <u>High-energy behaviour</u>:

Consider $\mathcal{A}(s,t)$ at $s,t \to \infty$ with s/t fix (fixed-angle high-energy scattering). Using $\Gamma(x) \sim e^{x \ln x}$ & "reflection formula", one sees that

$$\mathcal{A}_4 \sim \delta(\Sigma k_i) \exp\left(-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\right)$$

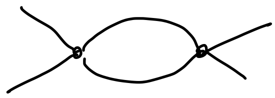
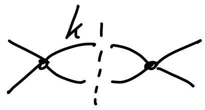
(away from poles).

This <u>"exponential suppression at high energies"</u> is very different from QFT or perturbative QG:

$\lambda\varphi^4$-theory: $\mathcal{A}_4 \sim \delta(\ldots)\lambda$; gravity: $\mathcal{A}_4 \sim \delta(\ldots)\frac{E^2}{M_p^2} \sim \delta(\ldots)\frac{s}{M_p^2}$

("E^2" homog. of degree 1 in s, t, u)

- From the QFT relation

 $\sim \int_k$ 

We may guess that the "soft" behaviour of amplitudes just discovered will also cure loop divergences. This does indeed happen.