15 Consistent superstring theories

15.1 GSO projection (Gliozzi, Scherk, Olive)

- Let F be an abstract "fermion number", defined on our Fock space.
- $(-1)^F$ has eigenvalues ± 1.
- $P \equiv \frac{1}{2}(1 + (-1)^F)$ is a projection operator ($P^2 = P$).
- Proposal: $\mathcal{H}_{new} = \mathrm{Im}\, P$ (This is consistent since $[P,H]=0$)
- F is defined by:
  - $(-1)^F |0,k\rangle = -|0,k\rangle$ (NS)
  - $(-1)^F |\alpha,k\rangle = |\beta,k\rangle \Gamma_\beta{}^\alpha$ (R)

    $(\Gamma \equiv \Gamma^{11} \equiv \Gamma^0\Gamma^1 \cdots \Gamma^9)$
  - $(-1)^F X^\mu = X^\mu(-1)^F$; $(-1)^F\psi^\mu = -\psi^\mu(-1)^F$

- Applying this projection to the open superstring gives:

  NS: no tachyon ; massless vector survives

  R: On our "Majorana spinor" vacuum, $P = \frac{1}{2}(1+\Gamma)$

  $\Rightarrow$ We are left with a Majorana–Weyl spinor with

  $$\underset{\text{go to "real"}}{2}\cdot 32 / \underset{\text{Maj.}}{2} / \underset{\text{Weyl}}{2} / \underset{\text{on-shell}}{2} = 8 \text{ real, on-shell d.o.f.}$$

- Observe: vector – 8 d.o.f. $\leftrightarrow$ spinor – 8 d.o.f.

  $\Rightarrow$ suspect 10d (target-space) SUSY!

- Indeed, there exists a very special and unique 10d SUSY QFT:

10d Super-Yang-Mills theory

Majora-Weyl fermion in adjoint rep.

$$S = \int d^{10}x \left(-\frac{1}{4} F^2 + \frac{i}{2} \bar{\Psi} \not{D} \Psi \right) \; ; \quad F_{\mu\nu} = -\frac{1}{ig} [D_\mu, D_\nu]$$

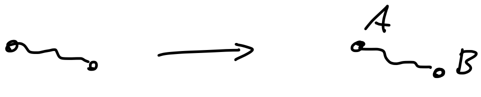
$$D_\mu = \partial_\mu - ig A_\mu$$

$$A_\mu \in \mathrm{Lie}(G)$$

Here: $G = U(N)$ (N D9 branes)

$\uparrow$

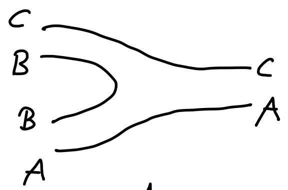
open strings come with "Chan-Paton-factors" specifying on which brane it starts and ends.

Chan-Paton in more detail: 

; $A, B \in \{1, \dots, N\}$

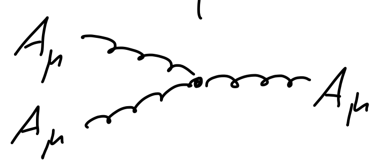
- If all b.c. are Neumann, this may be interpreted as the presence of N D9 branes on which strings can end. (More generally: Dp-branes.)

- Scattering requires matching Chan-Paton factors:



$\uparrow$

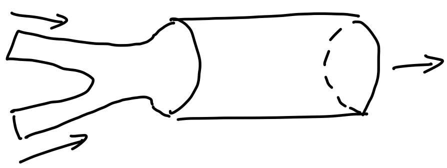
from YM-Feynman rule:

$\mathrm{tr}(T^a T^b T^c) \longrightarrow$ 

- General string state: $\lambda^{AB} |A, B, k\rangle$

Highly non-trivial: All interactions of SYM-theory come out right!

- Note: Open strings w/o closed strings are inconsistent since the latter can be created in scattering processes:



15.2 Type II superstring

(closed strings only, name comes from "10D N=2 SUSY".)

- Having understood the open string, which corresponds to having only a left- or only a right-moving sector, we now simply combine two such sectors, enforcing level matching:

- We label our building blocks by NS/R and +/− (for the eigenvalue of $(-1)^F$). We then have 4 sectors:

Sector	SO(8)-rep. of its lightest state		
NS −	1		tachyon
NS +	8_v (vector)		all massless
R −	$8'$	l.h. & r.h. spinors	all massless
R +	8	l.h. & r.h. spinors	all massless

- Recall that

$$L_0 = \frac{\alpha'}{4}p^2 + N + \phi \;\; ; \;\; \tilde{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} + \tilde{\phi}$$

with $\phi, \tilde{\phi} = 0$ (R) or $-1/2$ (NS)

- Enforcing level-matching, i.e. $L_0 - \tilde{L}_0 = 0$, we have:

$$\alpha' m^2 = 4(N+\phi) = 4(\tilde{N}+\tilde{\phi}).$$

- It is clear that $\alpha' m^2/4$ is integer for NS+, R+, R− and half-integer for NS− (starting with the tachyon at $-1/2$). Thus NS− can only be combined with itself while the other 3 can be combined in any way.

- This gives rise to the following (not yet GSO-projected) spectrum:

right, left		SO(8)-rep. of lightest state	
(NS−, NS−)	1	1	tachyon
(NS+, NS+)	9	$8_v \times 8_v$	all massless
(NS+, R−)	9	$8_v \times 8'$	all massless
(NS+, R+)	9	$8_v \times 8$	all massless
(R−, NS+)	9	$8' \times 8_v$	all massless
⋮		⋮	⋮

- In the open case, GSO selected 2 out of 4 sectors. Assuming we want to get rid of the tachyon, there was no qualitative choice.

- Now, we can select any subset of 10 sectors. These are 2^{10} possibilities.

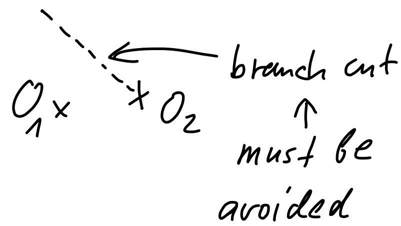
- We require one physical condition and 3 consistency requirements:

  1) No tachyon

  2) modular invariance  (any direction can be chosen as "time")

  3) mutual locality of vertex operators:

     (cuts come from $z^{1/2}$ for certain operators)



  4) closure of OPE

- The result of this analysis ($\to$ Polch.) is that only two selections work:

$\underline{\text{IIB}}$: $\quad (-1)^F = 1 \,, \quad (-1)^{\tilde F} = 1$

$$\text{IIA}: \quad \underbrace{(-1)^F = 1}_{\text{removes tachyon}} \,, \quad \underbrace{(-1)^{\tilde F} = \begin{cases} 1 & (NS) \\ -1 & (R) \end{cases}}_{}$$

$\leftarrow$ This means we choose $8'$, not 8.

Same (IIB) or opposite (IIA) chirality choice for spinor in R-sector of l.m.-side compared to r.m. side.

[exist IIA'/IIB' with $8 \leftrightarrow 8'$, but physically equivalent.]

- Alternative description:

IIB, r & IIA, r	:	NS+, R+
IIB, l	:	NS+, R+
IIA, l	:	NS+, R−

Resulting field content II A

Sector	SO(8)-rep.		decomposition in irreps.		
(NS+, NS+)	$8_v \times 8_v$	=	$[0]_\phi + [2]_{B_2} + (2)_G$	=	$1 + 28 + 35$
(NS+, R−)	$8_v \times 8'$	=	spinor $_\lambda$ + vector-spinor $_{\chi'_\mu}$	=	$8 + 56'$
(R+, NS+)	$8 \times 8_v$	=	spinor $_{\lambda'}$ + vector-spinor $_{\chi_\mu}$	=	$8' + 56$
(R+, R−)	$8 \times 8'$	=	$[1]_{C_1} + [3]_{C_3}$	=	$8_v + 56_t$

Notation: $[m]\,/\,(m)$ stand for antisymm. / symm. tensor of rank m.

χ_μ, χ'_μ are gravitini of opposite chirality

$$\left(N=2 \text{ SUSY} \Rightarrow \quad G_{\mu\nu} \begin{array}{l} \xrightarrow{Q_1} \chi_\mu \\ \xrightarrow[Q_2]{} \chi'_\mu \end{array} \right)$$

λ, λ' are "dilatini".

Counting of d.o.f.s:

- ϕ: 1 (obvious)
- "NS-2-form potential" or "Kalb-Ramond-field" $B_2 = (B_2)_{\mu\nu}\, dx^\mu \wedge dx^\nu$

  (as in bosonic case, $H_3 = dB_2$ is the corresp. field strength)

  As for photon, due to gauge-symmetry ($B_2 \to B_2 + d\chi_1$)

  only transverse components count: B_{ij} with $i,j \in \{2, .., D-1\}$

  $$\Rightarrow \# \text{d.o.f.} = \binom{D-2}{2} = \binom{8}{2} = 28$$

- Metric $G_{\mu\nu} \to G_{ij}$ (As above, but symmetric. In addition, tracelessness can be imposed.)

  $$\Rightarrow \# \text{d.o.f.} = \binom{D-2}{2} + (D-3) = \underbrace{\frac{D(D-3)}{2}}_{\text{generally useful for gravity in } D \text{ dims.}} = 35$$

- dilatino: λ, $\#$ d.o.f. $= 8$ (as explained for spinor in gauge th. above)

- gravitino $\chi_\mu \longrightarrow \chi_i$ $(i = 2, \dots, D-1)$ by gauge invariance, like for photon or graviton

  Additionally, the constraint $\gamma^\mu \chi_\mu = 0$ must be imposed, leading to a further restriction of the d.o.f. by "one spinor".

$$\Rightarrow \# \text{d.o.f.} = (\# \text{spinor d.o.f.}) \times (D-3) = 8 \cdot 7 = 56$$

- RR 3-form potential C_3: $(C_3) = (C_3)_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$; $F_4 = dC_3$

$$\# \text{d.o.f.} = \binom{D-2}{3} \leftarrow \text{only transverse relevant on-shell}$$

$$= \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

- RR 1-form potential C_1: like standard gauge th. $\Rightarrow \# \text{d.o.f.} = 8$

Resulting field content IIB

sector	SO(8)-rep.		decomposition in irreps.	
(NS+, NS+)	$(8_v \times 8_v)$	=	$[0]_\phi + [2]_{B_2} + (2)_G$	$= 1 + 28 + 35$
(NS+, R+)	$(8_v \times 8)$	=	spinor + vector-spinor	$= 8' + 56$
(R+, NS+)	$(8 \times 8_v)$	=	spinor + vector-spinor	$= 8' + 56$
(R+, R+)	(8×8)	=	$[0]_{C_0} + [2]_{C_2} + [4]_{+, C_4}$	$= 1 + 28 + 35_+$

Counting of d.o.f.

- Obviously, much of the analysis is as for IIA.

- A speciality is the appearance of the rep. $[4]_+$. Here the "+" means self-duality, i.e. $t_{i_1 \dots i_4} = (* t)_{i_1 \dots i_4} \equiv t^{j_1 \dots j_4} \varepsilon_{j_1 \dots j_4 i_1 \dots i_4}$ (with metric δ_{ij}).

- In the corresponding FT, self duality is imposed at the level of the field strength, i.e. $F_5 = dC_4$ with $F_5 = * F_5$

$\# \text{d.o.f.} = \frac{1}{2}\binom{8}{4} = 35$ ← using $\epsilon_{\mu_1 \dots \mu_{10}}$

Key lessons learned:

- Both theories have B_2 (allowing for the additional WS-action term $\int_\Sigma B_2$)
- Both have 2 gravitini and hence $N=2$ SUSY
- IIA is non-chiral (Both chiralities appear symmetrically)
- IIB is chiral (only one of the chiralities is realized by the gravitini).
- IIA/IIB have odd/even RR-forms C and hence the charged objects coming with them, the Dp-branes, are different:

IIA: odd RR-potentials $\Rightarrow \int_{D0} C_1 \,;\, \int_{D2} C_3 \,;\, \underbrace{\int_{D4} C_5 \,;\, \int_{D6} C_7 \,;\, \int_{D8} C_9}$

no new d.o.f. since

$F_p = * F_{d-p}$

(same d.o.f.; "C_{-1}" - no d.o.f.)

IIB: even RR-potentials $\Rightarrow \int_{D(-1)} C_0 \,;\, \int_{D1} C_2 \,;\, \int_{D3} C_4 \,;\, \int_{D5} C_6 \,;\, \int_{D7} C_8$

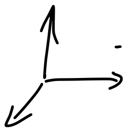
↑ "instanton" (point in spacetime)
(integral $\hat{=}$ evaluation of C_0 at point)

D9 plays special role - see below

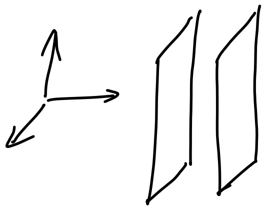
again, only half of forms directly visible in stringy d.o.f. since $F_p = * F_{d-p}$ (special: F_5 is self-dual and has only half of the naive # of d.o.f.: $F_5 = * F_5$.)

- We have introduced Dp branes as the charged objects under C_{p+1}, required by the consistency of 10d supergravity.

- They can or can not be part of a given geometry or background:



... empty 10d spacetime vs.

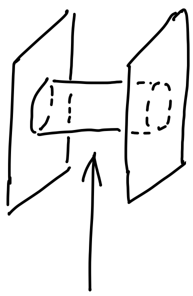


$\mathbb{R}^{1,3}$ with two (infinitely extended) D-branes

- These D-branes are <u>precisely</u> the objects on which open strings end.
- To see this, need to calculate coupling of RR-field to brane:

(Similarly, one can use the vertex operator for the graviton to determine the brane tension.) [We will not do these calculations.]

- A simpler, analogous calculation determines the brane-brane force from the "cylinder" diagram:



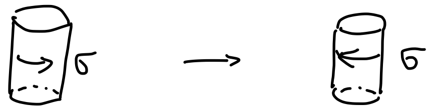
(cf. many textbooks)

closed-string exchange ≡ open-string one-loop diagram

- All of this eventually confirms that the D-branes sourcing RR-potentials are indeed the objects introduced earlier "on which open strings can end". They can be included in the 10d theory with their own action, the so-called DBI-action (which we will not discuss).

15.3 Type I superstrings

- Our WS theory on the cylinder has a $\mathbb{Z}_2$-symm, $\sigma \to -\sigma$, known as <u>orientation change</u>:



- Let Ω be the operator implementing this symm. in the quantized theory: $|\psi\rangle \to \Omega|\psi\rangle$. Obviously, $\Omega^2 = 1$ & $\Omega^{-1}\hat{X}(\tau,\sigma)\Omega = \hat{X}(\tau,-\sigma)$.

- After mode decomposition, one finds
$$\Omega^{-1}\alpha_n\Omega = \tilde{\alpha}_n$$
$$\Omega^{-1}\tilde{\alpha}_n\Omega = \alpha_n$$

- $P \equiv \frac{1}{2}(1+\Omega)$ is a projector, i.e. $P^2 = P$.

- We may restrict the theory to $\mathrm{Im}\, P$. (One also says "mod out Ω" or "gauge Ω" or "project the theory to the subspace invariant under Ω".)

- The result is the <u>unoriented string</u>.

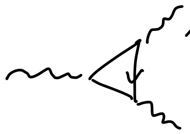
- Obviously, this projection can only be applied if left- & right-movers are treated symmetrically, i.e. <u>only in the IIB case</u>.

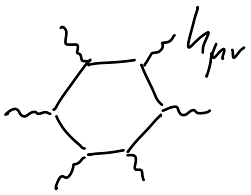
- Recall e.g. the NS, NS sector:
$$(8_v \times 8_v) = [0]_\phi + [2]_{B_2} + (2)_G = 1 + 28 + 35$$
$\uparrow$ states are $\psi^\mu_{-1/2}\tilde{\psi}^\nu_{-1/2}|0,k\rangle$, s.t. after the projection only the symmetric subrepresentations are left.

- A similar logic applies to the R, R & R, NS (NS, R) sectors, giving in total:
$$\phi,\ \cancel{B_{\mu\nu}},\ G_{\mu\nu},\ \cancel{C_0},\ C_2,\ \cancel{C_4},\ \chi^{(1)}_\mu,\ \cancel{\chi^{(2)}_\mu},\ \lambda^{(1)},\ \cancel{\lambda^{(2)}}$$

- Only $N=1$ SUSY is left!

- However, this projected or truncated theory is not consistent due to a gravitational anomaly.

  [This is analogous to the famous chiral $U(1)$ anomaly. The latter may be derived from current-non-conservation due to the diagram  . In the gravitational case, the external lines are gravitons. Also, the relevant polygon becomes larger in higher dims.: $\Rightarrow$ hexagon for $d=10$.

   .

  The symmetry at stake is diff, which is gauged. Hence a gravit. anomaly <u>must</u> be avoided.]

- The cure is unique and very simple: One has to add 32 D9 branes <u>before</u> the projection. As explained earlier, this corresponds to adding a $U(32)$ 10d SYM theory. The projection also acts on this theory and reduces it to an $SO(32)$ SYM theory.

  [One may superficially understand this by recalling that the gauge theory states are

$$\lambda^{AB} \, |A, B, k\rangle$$

$$\uparrow \; \uparrow$$

Chan-Paton-factors, specifying brane to which string is attached.

There is also an extra "$-$" associated with the action of Ω on the vacuum. Thus, together with the action $A \leftrightarrow B$, one is reduced from N^2 states (here $N=32$) to $(N^2-N)/2$ states, i.e. from the group $U(N)$ to $SO(N)$.

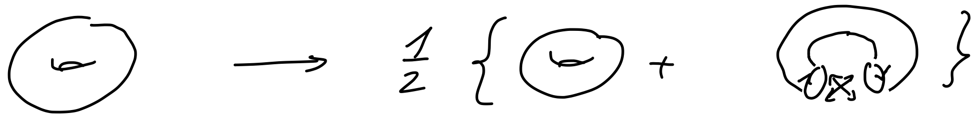
- Thus, we end up with a theory containing the surviving part of type IIB + an SO(32) SYM theory:

<u>Type I superstring</u>: $\left(1_\phi + 28_{C_2} + 35_{G}\right)_{Bos.} + \left(8' + 56\right)_{Ferm.}$

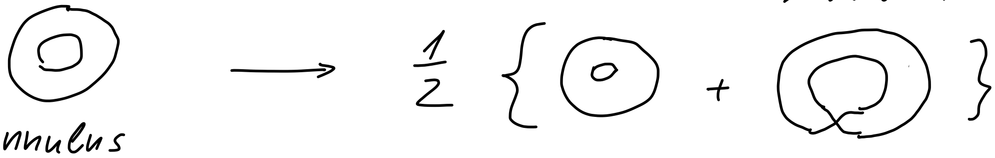
$$+ \; 496 \, (8_v + 8')$$

$\frac{N(N-1)}{2}$ ↗   Bos. ↑   Ferm. ↑   (gauge fields + gauginos)

<u>Note</u>: Loops in this theory are different in that one always must insert the projector $P = (1+\Omega)/2$ before closing the loop:



torus   glue with direction change $\Rightarrow$ Klein bottle



annulus   Möbius strip

<u>15.4 Heterotic string</u>

- On the WS, the left- & right-moving sectors are two independent theories.
- WS-SUSY together with the GSO projection exclude the tachyon

$\Downarrow$

Because of level matching, it is sufficient to supersymmetrize only one of the two sectors to get rid of the tachyon.

- Let us choose to keep the r.m. side supersymmetric, working also in $D = 10$:

<u>right</u>: $X_R^\mu(\sigma^-), \; \psi_-^\mu(\sigma^-) \qquad (\mu = 0, \cdots, 9).$

- Let us keep the l.m. side bosonic:

$$\underline{\text{left:}} \quad X_L^\mu(\sigma^-) \qquad (\mu = 0, \dots, 9)$$

- Recalling our previous results, the central charges are:

$$(c, \tilde{c}) = (\underbrace{10}_{X_R} + \underbrace{\tfrac{1}{2}\cdot 10}_{\psi_R} \underbrace{- 26}_{b_R c_R} \underbrace{+ 11}_{\beta_R\gamma_R}, \underbrace{10}_{X_L} \underbrace{- 26}_{b_L c_L}) = (0, -16)$$

- To ensure the key property $\tilde{c} = 0$, add 32 left-moving Majorana-Weyl fermions: $\lambda_+^A(\sigma^+)$; $A = 1, \dots, 32$

$$\Rightarrow S = -\frac{1}{4\pi}\int d^2\sigma \left[\frac{1}{\alpha'}(\partial X^\mu)(\partial X_\mu) - 2i\psi_-^\mu \partial_+ \psi_{-\mu} - 2i\sum_{A=1}^{32} \lambda_+^A \partial_- \lambda_+^A\right]$$

- The spectrum follows from mass-shell / level-matching conditions & GSO.
- On the r.m. side, we have as before: $R \to a = 0$; $NS \to a = -1/2$
- On the l.m. side we find, using known results:

$$R: \quad a = -8\,\frac{1}{24} + 32\cdot\frac{1}{24} = 1 \quad \Rightarrow \text{ no massless states}$$

$$NS: \quad a = -8\,\frac{1}{24} - 32\,\frac{1}{48} = -1 \quad \Rightarrow \text{ massless states arise from one bosonic or two fermionic excitations.}$$

- GSO can be chosen as in IIB:

$$(-1)^F = (-1)^{\tilde{F}} = 1$$

- Let us recall the IIB spectrum calculation in a shortened form:

$$\begin{array}{ll} \text{right:} & \text{left:} \\ (\underset{NS}{8_v} + \underset{R}{8}) \times & (\underset{NS}{8_v} + \underset{R}{8}) \end{array} = (8_v \times 8_v + 8_v \times 8 + 8 \times 8_v + 8 \times 8)$$

$$= \underbrace{1}_{\phi} + \underbrace{28}_{B_2} + \underbrace{35}_{G} + \underbrace{2(8' + 56)}_{\text{fermions}} + \underbrace{1 + 28 + 35_+}_{C\text{-forms}}$$

- The analogous calculation for the "SO(32) heterotic string" reads

$$(8_v + 8) \times ((8_v, 1) + (1, 496)) = \cdots$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ SO(8) & SO(32) & SO(8) & SO(32) \text{ rep.} \end{array}$$

$$\tilde{\alpha}^\mu_{-1}|0,k\rangle \qquad \underbrace{\lambda^A_{-1/2}\lambda^B_{-1/2}}|0,k\rangle$$

antisymmetric by fermionic nature of λ's.

$$\cdots = \underbrace{(1,1)_\phi + (28,1)_{B_2} + (35,1)_G + (56,1) + (8',1)}_{N=1 \text{ SUGRA}} + \underbrace{(8_v, 496) + (8, 496)}_{SO(32) \text{ SYM}}$$

fermions, gauge bosons, gauginos

[Note similarity to "Type I". In fact we are dealing with a different stringy UV completion of the same 10d field theory. Depending on ϕ, only one of the two is weakly coupled. This is called an S-duality ($S \hat{=}$ "strong-weak duality").]

- Our analysis was incomplete in that we tacitly assumed that the choice of NS vs. R must be made for all λ's at once. But this is not true. While most theories one might try to construct by choosing different b.c.s for different λ's are inconsistent, there exists a consistent theory based on splitting the λ's according to $32 = 16 + 16$ and allowing for different b.c.s for the two subsets.

- The naively surviving symm. group is $SO(16) \times SO(16)$ and the spectrum arises as (without derivation):

"$E_8 \times E_8$ heterotic string:"

$$(8_v + 8) \times \big((8_v, 1, 1) + (1, 120, 1) + (1, 1, 120) + (1, 128, 1) + (1, 1, 128) \big)$$

SO(8) SO(16) SO(16)' reps.

spinors of SO(16), SO(16)'
(arise in same way as SO(8) spinors)

Fact: $E_8 \supset SO(16)$

$$\underset{\text{(adjoint)}}{248} = \underset{\text{(adjoint)}}{120} + \underset{\text{(spinor)}}{128}$$

Fact: The resulting theory displays an enhanced gauge symmetry which is not just $SO(16) \times SO(16)$ but $E_8 \times E_8$

Summary: The 10d consistent superstring theories are:

I, IIA, IIB, heterotic SO(32), heterotic $E_8 \times E_8$.

This precisely matches all consistent 10d supergravities.