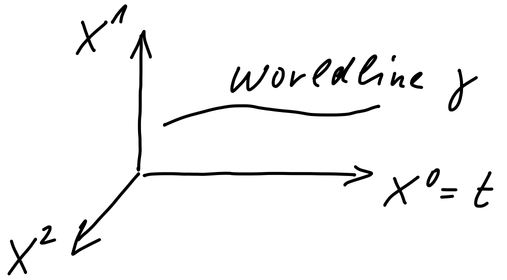
2 Classical Bosonic string

2.1 Relativistic point particle

The relativistic point particle is a very useful toy model, exhibiting some key aspects of the string. You should have already studied it in your course on special relativity.



} target space $\mathcal{M}$, coordinates X^μ
$\mu = 0, \dots, D-1$

The embedding of γ in $\mathcal{M}$ is specified by D fcts. $X^\mu(\tau)$, where τ parameterizes γ. The action, known from relativity, can be expressed in terms of these fcts.:

$$S = \text{"length of worldline"} = -m \int_\gamma ds = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$$

Here we used:
$$ds^2 = -\eta_{\mu\nu}\, dX^\mu dX^\nu \quad , \quad \text{i.e. } \mathcal{M} = \mathbb{R}^{1,D-1}$$
$$dX^\mu = \dot{X}^\mu d\tau . \qquad\qquad (\hbar = c = 1)$$

<u>You can check:</u>

- S is invariant under reparameterizations: $\tau \to \tau' = \tau'(\tau)$
- The EOM read $\ddot{X}^\mu = 0$
- The non-relativistic limit is $S = \int dt \left(\frac{m}{2} \vec{v}^{\,2} - m \right)$

- The action above is the point-particle analogue of the so-called "Nambu-Goto-action". We write: $S \equiv S_{NG}$.

- Both for point-particle and string, another useful action is the "Polyakov action" S_P, obtained as follows:

- Recall that on a manifold with coordinates y^a one measures distances using a metric: $ds^2 = g_{ab}\, dy^a dy^b$.

- Treat γ as a 1d manifold, with metric $ds^2 = h_{\tau\tau}\, d\tau^2$.
- A general action on γ would then be

$$S = \int d\tau \sqrt{-h}\, \mathcal{L}(X^\mu, \dot{X}^\mu).$$

- The specific choice

$$S_P = S_P[X, h] \equiv -\frac{m}{2} \int d\tau \sqrt{-h} \left(h^{\tau\tau} \frac{dX^\mu}{d\tau} \frac{dX_\mu}{d\tau} + 1 \right)$$

is called "the Polyakov action".

(Here $h \equiv$ "$\det h$" $= h_{\tau\tau}$; $h^{\tau\tau} = h_{\tau\tau}^{-1}$)

- One can check the following:

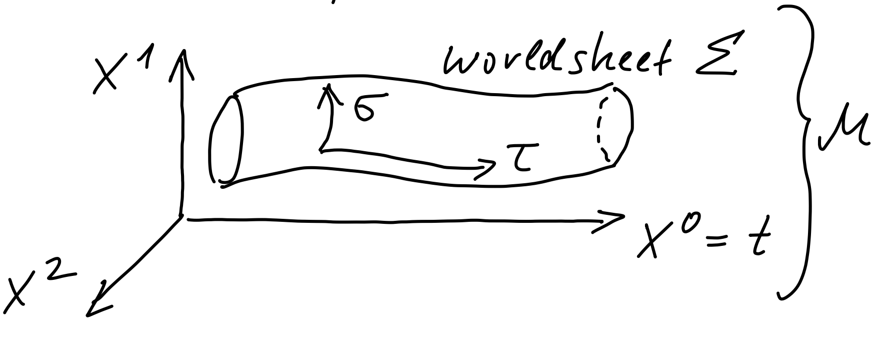
  - The EOM for h are: $\frac{\delta S_P}{\delta h} = 0 \Rightarrow h_{\tau\tau} = \dot{X}^\mu \dot{X}_\mu \equiv \dot{X}^2$

  - $S_P[X, h = \dot{X}^2] = S_{NG}$

$\Rightarrow$ S_P and S_{NG} are classically equivalent. S_P is much more convenient since it has no square root.

2.2 Bosonic string

Everything analogous:



- Embedding of worldsheet Σ in target space $\mathcal{M}$ specified by fcts. $X^\mu(\tau,\sigma)$.

$$S_{NG} = -T \int_\Sigma df$$

string tension $\nearrow$ (analogue of mass m)

$\underbrace{}$ area of Σ measured with target space metric $\eta_{\mu\nu}$

- It will be convenient to use a covariant coordinate notation also on Σ: $(\tau,\sigma) \equiv (\xi^0, \xi^1) \equiv \xi$

- An infinitesimal translation $d\xi$ on Σ induces an infinitesimal translation dX on $\mathcal{M}$, such that

$$ds^2 = -\eta_{\mu\nu}\, dX^\mu dX^\nu = -\eta_{\mu\nu}\left(\frac{\partial X^\mu}{\partial \xi^a} d\xi^a\right)\left(\frac{\partial X^\nu}{\partial \xi^b} d\xi^b\right) \equiv -G_{ab}\, d\xi^a d\xi^b.$$

- We see that $G_{ab} \equiv \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$ is the <u>induced metric on Σ</u>.

$$\Rightarrow \quad S_{NG} = -T \int_\Sigma d^2\xi \sqrt{-G} \quad ; \quad G \equiv \det G_{ab}$$

- In almost complete analogy to the point-particle case, we introduce an independent WS-metric h_{ab} and define the Polyakov action

$$S_P = -\frac{T}{2} \int_\Sigma d^2\xi \sqrt{-h}\, h^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

(Key difference: No constant term needed for classical equivalence with S_{NG}. We will show this in a moment.)

- Note: S_P is a field theory action for D free real scalars in two dimensions.

- A central object for such a theory is its energy momentum tensor, $$T_{ab} = \frac{4\pi}{\sqrt{-h}} \cdot \frac{\delta S_P}{\delta h^{ab}}$$

(This differs from the standard GR convention by a "stringy" normalization factor -2π.)

- We calculate: $$S_P = -\frac{T}{2}\int d^2\xi \sqrt{-h}\, h^{ab} G_{ab}$$

$$\delta(h^{ab}G_{ab}) = \delta h^{ab} G_{ab}$$

$$\delta\sqrt{-h} = -\frac{1}{2\sqrt{-h}}\delta(\det h) = -\frac{1}{2\sqrt{-h}}(\det h)\,\mathrm{tr}(h^{-1}\delta h)$$

identity for variation of any determinant

$$\mathrm{tr}(h^{-1}\delta h) = -\mathrm{tr}(h\,\delta h^{-1}) = -h_{ab}\,\delta h^{ab}$$

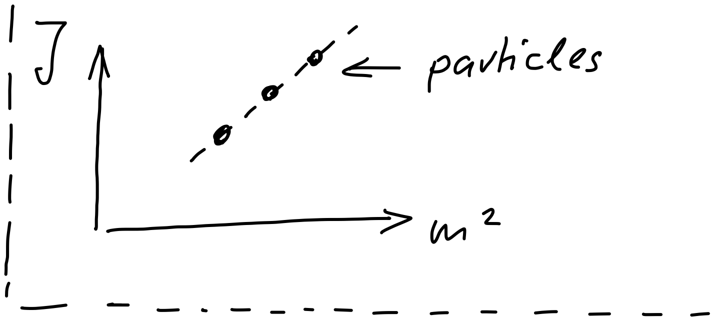
$$\Rightarrow T_{ab} = \frac{4\pi}{\sqrt{-h}}\cdot\left(-\frac{T}{2}\right)\left(\sqrt{-h}\,G_{ab} + h^{cd}G_{cd}\left(-\frac{h}{2\sqrt{-h}}\right)(-h_{ab})\right)$$

$$T_{ab} = -\underbrace{2\pi T}_{\equiv \frac{1}{\alpha'}}\left(G_{ab} - \frac{1}{2}h_{ab}\left(G_{cd}h^{cd}\right)\right)$$

with $\alpha' \equiv$ "Regge slope"

This name goes back to the time when string theory was invented as a model for hadronic physics:

hadron $\stackrel{?}{=}$  $\Rightarrow$ tension = prop. factor between m^2 & angul. mom.



slope of "Regge trajectory" $= \alpha'$

- The EOM for h is clearly $T_{ab} = 0$.
- This is solved by $h_{ab} = c\, G_{ab}$ for any fct. c:

$$\frac{1}{2} c G_{ab} \left(c^{-1} G^{cd} G_{cd} \right) = G_{ab} \quad \Rightarrow T_{ab} = 0$$

- $S_P[X, h_{ab} = c G_{ab}] = -\frac{T}{2} \int d^2\xi \sqrt{-c^2 G}\; c^{-1} G^{ab} G_{ab}$

$$= -T \int d^2\xi \sqrt{-G} = S_{NG} \quad \checkmark$$

2.3 EOM & Symmetries

$$S_P = -\frac{T}{2} \int d^2\xi \sqrt{-h} (\partial X)^2 \quad \text{with } (\partial X)^2 = h^{ab} (\partial_a X^\mu)(\partial_b X^\nu) \eta_{\mu\nu}$$

WS metric

metric on "field space" of our 2d QFT

Symmetries:

1) Diffeomorphisms: $\xi^a \longrightarrow \xi'^a = \xi'^a(\xi^0, \xi^1)$

2) D-dim. Poincare-invariance:

$$X^\mu \longrightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu + V^\mu \; ; \quad \Lambda \in SO(1, D-1)$$

(This is an internal global symm. of our 2d QFT.)

3) Weyl-rescaling invariance:

$$h_{ab}(\xi) \longrightarrow h'_{ab}(\xi) = \varphi(\xi)\, h_{ab}(\xi).$$

The fact that such a rescaling factor $\varphi(\xi)$ drops out of the action is a key special feature of $\underline{\underline{d=2}}$.

- The EOM are: $h \rightarrow T_{ab} = 0$ (see above)

  $X^\mu \rightarrow \Box X^\mu = 0$ (standard QFT result)

  $\Box \equiv D_a \partial^a$

- Comment 1: As in GR, diffeomorphism invariance implies $D_a T^{ab} = 0$ (even before the EOM of h sets T^{ab} to zero).

- Comment 2: $T_a{}^a = 0$ holds as an identity – without using EOMs (<u>Problem</u>: Derive this from the symmetries of S!)

<u>2.4 Choosing a gauge</u>

Both "Diff.s" and "Weyl rescalings" do not affect the embedding of Σ in $\mathcal{M}$. $\Rightarrow$ We declare them to be <u>gauge symms.</u>, i.e. they relate different descriptions of the same physics.

- <u>Key claim:</u> Using Diff. & Weyl, we can locally ensure

  $$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ This is called "flat gauge".

- <u>Naive argument:</u>

$$\left.\begin{aligned} \xi^a &\rightarrow \xi'^a(\xi^0, \xi^1) \\ h_{ab} &\rightarrow h_{ab} \cdot \varphi(\xi^0, \xi^1) \end{aligned}\right\} \text{3 arbitrary fcts.}$$

Since h_{ab} contains only 3 arbitrary fcts., we generically have enough freedom to bring h_{ab} to any desired form.

More precise argument:

- Consider the Ricci scalar of the WS with metric h_{ab}: $R[h]$. A straightforward calculation shows that (see e.g. Wald):

$$h'_{ab} = e^{2\omega} h_{ab} \quad \Rightarrow \quad R[h'] = e^{-2\omega}\left(R[h] - 2D^2\omega\right).$$

- Given some metric h, we can now solve the PDE $D^2\omega = R[h]/2$ for ω. This is a simple wave equation with a source. On a cylinder, with some cut through the cylinder as a Cauchy-surface, this will always have a solution. Having found ω, we rescale $h_{ab} \to h'_{ab} = e^{2\omega} h_{ab}$. Now we have gauge-equivalent metric h' with $R[h'] = 0$.

- Specifically in $d=2$, we have $R_{abcd} = \frac{1}{2}[h_{ab}h_{cd} - h_{ad}h_{bc}]\cdot R$. Thus, our new metric has vanishing Riemann tensor and is hence flat. In other words: On choose coordinates s.t. $h'_{ab} = \mathrm{diag}(-1,1)$.
- More general than this "flat gauge" are conformal gauges, where h_{ab} is only flat up to a rescaling $h_{ab} \to e^{2\omega} h_{ab}$.

Comment:

We will later consider the euclidean version of our 2d theory. Then Σ's other than torus (or strip) will become relevant and the existence and uniqueness of a flat gauge choice will become highly non-trivial and important. See BLT, Sec. 2.3 & 6.2 for more powerful methods useful in this context.

2.5 Solutions

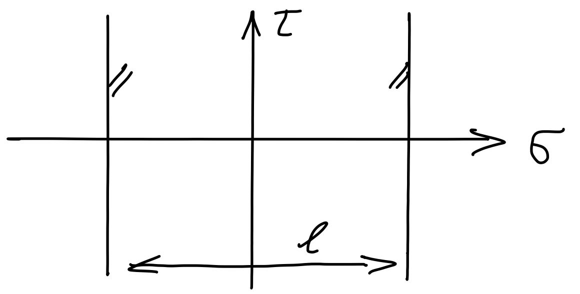
- We use flat gauge and light-cone coordinates: $\sigma^{\pm} \equiv \tau \pm \sigma$.

$\Rightarrow$ $ds^2 \equiv -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-$, i.e.

$$h_{++} = h_{--} = 0 \ \& \ h_{+-} = h_{-+} = -\tfrac{1}{2} \ \& \ h^{+-} = h^{-+} = -2$$

$$\Box = h^{ab}\partial_a\partial_b = 2h^{+-}\partial_+\partial_- = -4\partial_+\partial_- \quad \text{with } \partial_{\pm} \equiv \frac{\partial}{\partial\sigma^{\pm}}$$

- EOM: $\partial_+\partial_- X^{\mu} = 0$
- Any solution can be written as: $X^{\mu} = X_L^{\mu}(\sigma^+) + X_R^{\mu}(\sigma^-)$
- The index L/R stands for left-/right-moving wave, explained by the parametrization of the cylinder as



$\Rightarrow$ $X^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma+\ell)$

(By diff-invariance, we can choose any desired value for ℓ.)

- X^{μ} periodic in σ $\Rightarrow$ $\partial_+ X^{\mu} = \partial_+ X_L^{\mu}$ & $\partial_- X^{\mu} = \partial_- X_R^{\mu}$

  both periodic in σ.

$\Rightarrow$ $\partial_+ X_L^{\mu}$ & $\partial_- X_R^{\mu}$ can be written as $\sum_{n\in\mathbb{Z}} e^{-2\pi i n\sigma^{\pm}/\ell}$

$\Rightarrow$ X_L^{μ} & X_R^{μ} follow by integration and hence, in addition, contain a linear term.

$\Rightarrow$ General solution:
$$X_L^{\mu} = \tfrac{1}{2}x^{\mu} + \frac{\pi\alpha'}{\ell}p^{\mu}\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{\mu}e^{-2\pi i n\sigma^+/\ell}$$
$$X_R^{\mu} = \tfrac{1}{2}x^{\mu} + \frac{\pi\alpha'}{\ell}p^{\mu}\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-2\pi i n\sigma^-/\ell}$$

Comments:

- The constant (x^μ) is the same in $X^\mu_{L,R}$ by convention.

- The coefficient p^μ of the linear term must be the same for periodicity of X^μ in σ.
- X^μ real $\Rightarrow$ x^μ, p^μ real & $(\overset{(\sim)}{\alpha}{}^\mu_n)^* = \overset{(\sim)}{\alpha}{}^\mu_{-n}$.

- $X^\mu = x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau + \dots$ = liner motion + fluctuations.

- By Diff+Weyl invariance, our choice of ℓ is arbitrary. Moreover, the coeffs. in the "oscillator expansion" above are conventional. We gave the form of BLT. One can for example also follow GSW (cf. also my old notes) and choose $\ell = \pi$, and in addition set $\alpha' \equiv l_s^2/2$, with l_s the "string length".

$$\Rightarrow \quad X_L^\mu = \frac{1}{2} x^\mu + \frac{l_s^2}{2} p^\mu \sigma^+ + \frac{i l_s}{2} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$

$$X_R^\mu = \frac{1}{2} x^\mu + \frac{l_s^2}{2} p^\mu \sigma^- + \frac{i l_s}{2} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-},$$

cf. also my old notes. (Do not confuse ℓ & l_s - they are conceptually different quantities.)